Mathematical Modeling: Teaching the Open-ended Application of Mathematics

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To my parents, who are wonderful models for how to live a life.

To my high school teachers Stewart Galanor and Antonia Stone, who taught me well and then put me to work.

To my wife, Emily, for all of her encouragement.

<u>Numbers in Context</u>

Student Skills	• Note taking (see Coaching in the Pedagogy chapter). • Writing. • Setting standards for evaluating work. • Working effectively in groups. • Memorizing.

Content	**Modeling Skills and Habits**
Scale Models and Comparisons	*See the unit for details.*
Technology and Magnitude • Calculator use, calculator number representations.	• Testing Smaller Cases/Simplifying. • Extending results from simplified settings. • Understanding the limits and uses of technology. • Making informed judgments about appropriate tools. • Checking results. • Writing explanations of observations and discoveries.
Estimation and Number Magnitude Instincts • Familiarity and computation with large and small quantities. • Associating units with numbers. • Benchmark facts (populations, etc.). • Precision vs. accuracy.	• A tolerance for imprecision. • Using units. • Identifying needed information. • Researching information. • Choosing appropriate mathematical tools. • Testing results for consistency. • Understanding and supporting group efforts. • Defining terms.
Number Systems and Properties • Number types and representations. • Dense and closed sets.	• Understanding definitions. • Testing cases. • Generating examples and counterexamples. • Justifying conclusions. • Introducing variables to represent a quantity. • Proof.
Cantor and Infinity	• Proof that actually changes your mind. Being skeptical about common sense intuitions. • Beautiful math for its own sake.

Unit Overview

Descriptive numbers accompany real world questions. This unit helps students think about, and work with, numbers in context. For a student to work successfully with these numbers, they must develop a range of skills and a broad knowledge base. Numbers in context can be quite large or quite small, tend to have units, and are often not known with complete precision.

Numbers serve as an appropriate entry point for many of the objectives of this course because, while they are familiar and tangible, they can lead us into surprising and sophisticated territory. The unit begins with a sequence developing students' understanding of extreme quantities, estimation, units, accuracy and precision, and number types. This segment's emphasis on reading, writing, and inexactness helps to reinforce the message of the goals sheet (see **I1.1** and Objectives in Chapter 1) that this course will expect thinking that differs from the students' conceptions of typical mathematical work. It also develops several skills that are prerequisite to mathematical investigations of real world issues.

The first written project, creating an "exponential ladder" of examples of one unit of measurement, familiarizes the students with the different scientific realms and the ranges of quantities that they involve. It also is an application of exponential growth that helps develop an intuitive understanding of the difference between exponential and linear behavior. As the explorations move in more theoretical directions the themes of proof and the infinite recur.

Quantity and shape are two of the fundamental objects of mathematical study. Geometry courses frequently assume that students can visualize three-dimensional objects based on two-dimensional representations and can recognize figures imbedded in more complicated drawings and may not explicitly teach students these skills. Algebra courses make similar false assumptions about number comprehension. As mathematics curricula increasingly emphasize application, modeling, and technology (both as a tool and as an object of study), we must explicitly develop number sense in our students. The NCTM Standards 2000 make number sense and estimation high-ranking objectives throughout the K-12 curriculum. When they have been taught previously, these skills need to be reinforced and extended at the secondary level. In many cases, high school students have not yet experienced curricula rich in these topics.

The number sense that I focus on here includes: vocabulary (names for numbers, "order of magnitude", unit names and their definitions), computation skills (scientific notation, calculator use, comparison abilities), and methods of estimation. Students must have an intuitive feel for quantities that are more extreme than those encountered on a daily basis. Students need considerable practice with scientific notation (taught in middle schools) and should be able to use it to make computations such as: $5 trillion national debt ÷ 250 million people in the population = $20,000 of debt per person. Scientific notation's benefits derive from its compactness and the fact that it draws our attention to the exponent (or magnitude) of a number. It is also a format that we use for communicating with our calculators.

Number sense is crucial to understanding the limitations of the tools that we use. This understanding has always been important for measurement and remains so with calculators and computers. While number sense is crucial to the appropriate use of technology, technology in turn has greatly expanded our access to efficient work with extreme quantities. Mathematical models naturally involve small and large quantities. For example, alcohol levels in the blood stream drop at a rate of 0.015% an hour. Functions with constants such as this are typically encountered when studying messy, real world data and it is in such cases that we discover our students' real understanding of numbers.

Scale Models and Comparisons

Activity: *Building a Scale Model of the Solar System*

Technical Skills: Setting up and solving proportions; converting units; measurement; various geometry ideas depending on the methods which students choose.

Modeling Skills: Choosing appropriate mathematical tools; identifying needed information; researching information; decision making in groups; testing results for consistency[1].

Audience: Middle school onward. This task is thought provoking even for advanced students.[2] It is a fun and engaging introduction to several strands of the course.

Materials: An inaccessible round object (e.g., the ball on the top of a flag pole although a larger sphere is preferable), a variety of very small to medium-sized spheres, several yard or meter sticks, handouts **N1.1** and **N1.2**. Students will need notebooks and calculators.

Length: Two hours or more depending on the number of planets laid out. Can be done over two class periods, in which case the homework might be to research any information that will be needed during the second session.

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1 Because the setting is tangible and because there are several possible relationships that students may consider (e.g., comparing the radii of, and distances between, the planets and Sun), students are more likely to try to make comparisons than with a more abstract problem.

At the start of the activity, students assemble around an inaccessible round object. My colleagues and I have used a large stone sphere perched roughly 20 feet up on a stone pedestal in a nearby park (see the picture below). Students are asked to determine where the Earth and other planets would be located in the park, if the stone sphere were the Sun in a scale model of our Solar System (**N1.1**). This interdisciplinary activity can serve both the math and science curriculum if teachers coordinate their focus on measurement and quantity.[3]

In groups of three, students attempt indirect measurements of the diameter of the sphere. The positions of the Sun and clouds influence just how difficult this task is each year. Students who notice a shadow of the sphere discuss whether it is the same size, smaller, or larger than the sphere itself. Because the Sun's rays are essentially parallel at this distance, the shadow has the same width as the stone and its length depends on the Sun's angle. Most ignore the shadow and attempt to "eye-ball" the diameter by comparing it to other objects or by setting up proportions with varying degrees of complexity and sophistication to their method. The teachers mull around asking questions of the groups to make sure that they are working as groups and that all of the members of a group agree on their conclusions. If there is a dissenter, we will encourage the group members to clearly state their assumptions and plans so that the differing ideas can be critiqued and greater consensus can be reached.

Once students have estimated the diameter, they realize that they need information on the Sun and planets (not coincidentally in the possession of their teachers). Many seem puzzled about how to proceed and only those who ask us point blank about the information that they need are provided with it. The sheet (**N1.2**) that they get has far more information than they require and is

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2 In fact, I frequently find that the students who have taken the greatest number of traditional math classes struggle the most with this activity. Their ability to grind through well-defined exercises that has produced such success in past courses makes them the most disconcerted when the "rules" of math class change.

absent one useful value. The missing information is the moon's diameter. Absent any help, students begin to see each other as a resource and puzzle out an estimate together.[4] We play these "games" because we want the students to start realizing that the active acquisition of relevant information is an important stage in any investigation. Students are too used to being given exactly the information they need to solve a problem. In real situations, information is usually too scarce or overly abundant.

Students calculate the scale model sizes and distances for each planet. The teachers check in with each group to see how the values are being calculated. We are interested in whether the problems are being treated as numerous unrelated proportions or a sequence of scalings all by the same factor. Either approach can work, but the latter is more efficient and reveals greater attention to pattern and structure (e.g., do they realize that they are dividing by the Sun's diameter in each proportion?). This distinction is discussed in class the next day.

Once the groups have completed their calculations, they discover that only the inner planets even fit in the park. Their earth object should be a little more than 100 meters away. Model Pluto would be roughly $2^1/_4$ miles from the model Sun and its orbit would include much of the city.[5] It is fun to watch the different ways students march off the distances for the Earth. These differences drive home how the same task, measuring a length, is really several different tasks when new settings or scales are introduced. Measuring a pencil's length with a ruler is simpler than marking off 110 meters with your feet or a yardstick. Both of these measurements are far easier than determining the inaccessible lengths that constitute the solar system data itself.

The groups wade into a box of spherical objects (pin heads, small foam balls, sports balls, balloons, inflatable globes, etc.) and identify objects that would be the correct size for the planets relative to the approximately meter wide stone "Sun". Once they lay out these objects in the park at the calculated scale distances, they are asked how they can check their results. One check the students frequently make as they lay out the planets is to see if a planet is too close to the column holding the sphere. They realize that the orbital distances should be much greater than the width

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3 William Luzader, who did much to enrich my understanding of astronomy and physics, introduced me to this activity.

4 If they are really stuck (and it is fine if they do not figure out 3476 km exactly!), you might ask them how a knowledge of eclipses might help (because the sun and moon have the same approximate apparent diameter, comparisons can be made).

5 These values will differ, of course, according to the size of your model Sun. If you find the ratio of your model Sun's radius to the real Sun's radius (or Semi-diameter), you can multiply all needed values from N1.2 by this scale factor.

of the Sun, so if they get a scaled answer of 2 feet for a planet's placement, they know there is a problem.

There is a beautiful method for checking their success, and once groups have had a chance to come up with their own approach, we do present it to them. If they hold a pinkie up at arm's length, its width should approximately cover the Sun in the sky. If they stand at their model Earth, their outstretched pinkie should cover the same amount of the stone sphere. That comparison highlights what it means to have made a scale model! The ratios are maintained and the stone sphere should have the same apparent diameter in the sky as the real Sun does. The students are simultaneously in the same place (on the Earth) in two different scale versions (one real). If their pinky covers one Sun more than the other then they are asked to use that information to adjust the position of their model earth and to check again. They should explain why they are moving farther in or out in response to their first check. We also ask them what this adjustment tells them about their estimate for the size of the stone sphere (if they underestimated, their model will be too small, the sphere will appear too large relative to their pinky, and they will have to move the Earth farther away). The ability to physically check their results highlights the benefit of working with a model instead of just the list of scaled numbers. A model can be more readily explored and checked against reality and prior knowledge.

The activity culminates with the students walking around to the other model planets that they have placed in the park. There they look at the stone sphere and see how big the Sun would appear in the sky if they visited those worlds. I try, with no difficulty, to convey my excitement that they have built a scale model sufficiently large enough to experience this comparison. In contrast, the model Earth to Moon distance (≈ 1 foot) is small relative to the size of the students, so it is difficult to check to see if the Moon's apparent diameter is correct.

Although students cannot march off the two miles to Pluto, they can imagine what would be trickier to picture without this activity. They can determine how the Sun appears from Pluto by considering what the sphere would look like from two miles away. The answer is that it would be imperceptible unless, like the Sun, it was illuminated. Thus, for someone on Pluto, the Sun would appear as the brightest star in a perpetual night sky. Posters and models of the solar system that I have seen display orbital lengths and diameters in separate scales making this vastness of the solar system less obvious.

A hydrogen atom consists of a single proton in its nucleus orbited by a single electron. For homework, students are asked to determine where and how big the electron would be if the stone sphere in the park were the nucleus of a hydrogen atom. This assignment again requires the identification of needed information and the formulation of appropriate proportions and/or scaling factors. Depending on the definitions and methods used, the diameter of a proton ranges from 10^{-15} to 10^{-13} meters and the electron orbits 10^{-10} meters away (1 Angstrom (Å)). The greatest possible answer with the electron 100 km from the sphere (well beyond model Pluto's orbit) reveals the extraordinary emptiness of the atom. During homework discussions, I ask the class whether the solar system or an atom is denser. Many choose an atom, but someone usually points out that the solar system is a collection of atoms separated by emptiness, so it must be less dense than the atoms themselves. This discussion highlights one way in which scale models can deceive us.

The size of the electron in the hydrogen atom model is a tricky question. Electrons have different diameters depending on the nature of the measurement attempted and it may be difficult to find an estimate. Students might want to consider the relative masses of the proton and electron as an informal guide. They would need to decide whether mass relates to volume and how volume relates to diameter (not proportionately).

<u>Teaching tips for this activity</u>

I always struggle to resist the temptation to tell students how to do a task. Listen to groups as they work and, if they seem to need help, ask questions that you would ask and answer if you were at the same point in the process. The estimate for the length of this activity will give most groups sufficient time to grapple with the ideas. Some get bogged down estimating the sphere's size; others struggle to agree about the relevant mathematical approach; others make mistakes calculating the scaling factor and ratios. Without providing too much guidance or structure, there may be groups that you want to encourage to arrive at a simplest approach to their current dilemma so that they may move on if they are way behind the rest of the class. Try to do so only if you feel their discussions are unproductive. If they are stuck, ask them what questions they have, what assumptions they are making, or where their disagreements are. Once they can identify these sticking points for you, let them return to work on them themselves (and explain that you are doing so). The act of naming a problem can sometimes provide the clarity that is necessary to productively return to a problem-solving effort.

For older students who will be studying, or have already studied, radian measure and trigonometry, the pinky method provides a vivid example of these ideas at work that can be referred to at an appropriate time. A pinky held out at arm's length covers roughly 1° of arc. The Sun covers a little more than half of a degree. Students can find both of these values using the inverse tangent function. The angle can also be computed in radian measure using $\theta = s/r$ = arc length/radius = pinky width/arm length which for my arm is roughly 1 cm /70 cm or 0.14^R. Similar computations for the Sun will produce a smaller angle. We can emphasize the relationship between radian measure and proportionality by comparing lengths and angles within the model and the real solar system.

Lastly, if your students are familiar with the formula for the surface area of a sphere or inverse square functions ($y = k/x^2$) or if you want them to be, you can ask about the relative brightness of the Sun from each of the planets. Because the constant amount of light emitted from the Sun during a time interval spreads out over ever-larger spherically shaped shells whose area is $4\pi r^2$, the number of photons per unit area decreases with the square of the distance from the Sun. For example, if the apparent brightness of the Sun as seen from the Earth is 1, then the apparent brightness from Jupiter, which is roughly 5 times farther away, would be $^1/_{25}$ (as long as you ignore atmospheric effects).

Technology and Magnitude

Many elements of the representation stage of the modeling cycle (see **P1.1**) can be discussed and practiced individually. The messiness of real world problems, however, can complicate the process of learning these steps. It is sometimes more effective if we emphasize key ideas and questions by beginning with problems which lack the complexity of real world contexts. After these introductory explorations, students must have repeated opportunities to use their skills further with less-structured problems. These opportunities increase the likelihood that students can determine when and how to use each skill. Teaching skills in isolation is only useful if we provide the support for students to transfer their learning to the real problems that they will face.

The second homework assignment of this unit is a handout on calculator precision and rounding (**N2.1**). It is important that students be aware of the limitations and behavior of the technology that they use. These exercises force students to anticipate and then think about calculator results before accepting them. One problem on the sheet, $10^{50} - 10^{20}$ which yields a calculator result of 10^{50}, is an example of how a single exercise can produce a wealth of learning if student reasoning is fully explored. Remarkably, some students do not question the answer. Others are shocked by the calculator's "stupidity". They know the answer cannot remain 10^{50} after subtraction. When asked for the true answer, they frequently shrug or suggest 10^{30}. When asked for the rule that matches their action ($a^x - a^y = a^{x-y}$), they usually realize that it is not a rule that they have studied and that division and canceling are what produce subtraction of exponents. Students often have difficulty producing the correct value for $10^{50} - 10^{20}$ because they stay with exponential notation.

Students' difficulty solving the above problem, which is nothing more than integer subtraction, highlights the importance of considering how students transfer what they have learned and how much they recognize the limitations of techniques which they have been taught. Students, who successfully apply a concept (such as a property of exponents) in a situation similar to those in which the concept was learned, may still be unable to generalize that understanding. Students know how to measure lengths, but during the solar system scale model activity, they are uncertain how to measure off 100 meters. Many of our number and measurement skills apply only to "everyday" ranges and many other skills are inaccessible to students when not learned in a variety of open-ended settings.

The $10^{50} - 10^{20}$ discussions serve many purposes. They help students relate abstract algebraic rules to concrete numerical examples; they make students appropriately suspicious of their calculators; and they highlight the fundamental challenge modelers face interpreting a simplified representation of an idea.

Students often have difficulty producing the correct value for $10^{50} - 10^{20}$ because they stick with the exponential notation. Some try writing out the two numerals and carrying out the borrowing to find out how many *9*'s and *0*'s will be in the answer:

$$\begin{array}{r} 100{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000 \\ -\quad 100{,}000{,}000{,}000{,}000{,}000{,}000 \\ \hline \end{array}$$

The above subtraction can work, but it is cumbersome and error-prone when students write it out by hand. Others think to explore a smaller case such as $10^5 - 10^2 = 100{,}000 - 100 = 99{,}900$. Sometimes students claim that, since the exponents in their smaller example are ten times smaller than in the original problem, they need to scale the answer up with ten times as many *9*'s and *0*'s (so the answer should be thirty *9*'s and twenty *0*'s). When a generally useful strategy, such as trying smaller cases, is employed, it should be highlighted, applauded, and discussed further. This habit of reducing the complexity of a problem in order to gain understanding is an essential step in creating mathematical models (as well as a powerful problem solving tool in pure mathematics). The $10^{50} - 10^{20}$ problem provides an uncomplicated setting for raising two questions that must be addressed when simplifying and abstracting a problem:

<u>In what ways can we make the situation smaller or simpler?</u> What counts as "smaller", and which approach will be most revealing, is frequently not obvious. Should we calculate $10^{50} - 10^0$, $10^5 - 10^2$, or some other variation? For applied problems, modelers may change a discrete variable to a continuous one, substitute a predictable behavior for a random one, or choose a simpler shape or smaller dimension. For example, a group studying the scheduling of maids at a motel began with guests who arrived at predetermined times and only later allowed for random arrivals. The political parties model made the distances between adjacent views uniform despite the fact that the gaps might be irregular.

<u>Once we have a result from a smaller case, how do we apply our understandings of it to the original case?</u> For example, 99,900 has three *9*'s and two *0*'s. Can we multiply both counts by ten as students did? Some students note that the second exponent is the number of 0's and the difference between the exponents (5 – 2) is the number of 9's. Does this observation explain the first claim? Is either approach always correct? Interestingly, both versions work, but they must be tested and explained for more than one small case before they can be extended confidently. A big difference between the $10^{50} - 10^{20}$ problem and the modeling examples is that the simpler arithmetic case still contains most of the richness of the initial problem, whereas models of real world situations are often far less

complex than the original setting. Modelers must often jettison important variables before a workable model can be created. The task of extrapolating the results back to the real setting is then more fraught with uncertainties. The group modeling lobster harvesting was concerned with unpredictable fluctuations in the lobster population (see About This Book in the Pedagogy chapter), but their model was deterministic (it behaved the same way each time, it did not include any random variables). This choice meant that they had to add an additional margin for error to their recommended harvest totals.

Whenever students create a representation of a setting, they should return to the two underlined questions above, weigh their options, and justify their choices. The utility and limits of a model will be best understood when these questions are thoughtfully answered. Highlight for students that the first question is part of the process of representation and the second question arises during the analysis stage. However, the modeling cycle is not as clean and linear as pictured in the diagram (**P1.1)**. Modelers consider how their conclusions will be affected by the simplifications that they choose. Anticipated answers to the second question above affect which of the first question answers they pursue. This looking ahead occurs at every stage of modeling, so there should really be arrows all over the diagram. As students study and create a variety of models, they will grow in their ability to make informed choices at each step.

Once the correct answer to the $10^{50} - 10^{20}$ problem is found, the calculator answer must still be explained. Scientific calculators typically store around a dozen decimal places for each calculated result. Displayed answers are then rounded to the closest ten decimal place real number. Note that 10^{50} is far closer to the true answer (being off by "only" 10^{20}) than is the truncated answer 9.999999999×10^{49} (which is off by almost 10^{40}). Another explanation, which draws on an understanding of magnitude, was offered by one of my students who pointed out "10^{20} was small in the scheme of things!" This was an accurate claim given the "scheme" presented. Students can also be asked how many times bigger 10^{50} is than 10^{20} (answer: 10^{30}). Students who have found the true difference between the two may still respond with "$2^1/_2$ times" or "10^{20} is 40% of 10^{50}." They should be asked how they arrived at their answer and how they would perform a similar computation for two numbers not in exponential form. A concrete comparison is that a single molecule of water is a greater part of a tub of water than is 10^{20} a part of 10^{50}.

For all answers to handout **N2.1**, be sure that students do not record results in the form "1.5 <space> 12". They need to convert these calculator displays into the scientific notation equivalent, 1.5×10^{12}. The errors that appear in these exercises are due to the finite number of decimal places stored for calculation results that have lengthy or infinite decimal representations. Calculations A, C, F, G (for some calculators), H, and I produce rounded or truncated answers:

A) $2\div3$ has an infinite expansion that cannot be fully written out.

C) The last digit of this multiplication should be 4 ($8 \times 8=6\mathbf{4}$).

F) 99999999999999999999999999999900000000000000000000. Some students get 10^{51} as an answer because they enter 10 EE 50 rather than 1 EE 50 to represent 1×10^{50}.

G) Some calculators produce very small negative answers because $1 \div 3$ is represented by .333333333333, which is a little less than one third. If you type $1 \div 3 \times 3$ first, a TI-8X will show 1, but if you type "-1" (producing "ANS - 1"), the result will not be 0. These steps demonstrate that the calculator's display is a rounded version of the value held in memory. A similar error occurs with log(16)/log(2). This calculation correctly displays 4, but the int() or ipart() of it yields 3 as an answer.

H) Look at 1/9, 1/99, 1/999, etc. to see what your answer should be. The "try a simpler case" strategy can be celebrated here, too, if a student presents it.

I) Same as F, only at lower magnitudes.

Estimation and Number Magnitude Understanding

Units and Conversion

A main goal of this course is that we focus our teaching on the skills of citizenship within a democracy. Citizens need to be able to read newspapers and make sense of the quantities and claims they encounter there. For example, when the AIDS epidemic first began, there were calls for universal HIV testing. Such an action has important economic, legal, and medical consequences. One consequence is that the test for HIV is not perfect and an enormous number of people would be falsely informed that they were HIV positive. Simplistic solutions to social and political problems have great power when people passively consume information.[6]

Our students must learn to actively question what they read and contrast it with what they know. Helping them become alert to the different numbers that they encounter is an important first step. Important, yet difficult to comprehend, numbers appear in the media regularly (e.g., our world population exceeding 6 billion people). Nonsensical, but often accepted, numbers appear with disturbing frequency as well. Often, a writer has mistaken one "-illion" for another. A few years back, a New York Times photograph of the newly renovated Ellis Island had a caption

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[6] The "Star Wars" missile defense system, which proponents claim will shield the United States from enemy attacks, is a similarly simple sounding and appealing idea. Unfortunately, it would take over a hundred billion dollars to build and trial tests of missile interceptors regularly fail even under optimal conditions. Scientific American has published articles detailing the technical reasons (from the fields of physics and computer science) why such a system is easily defeated. The political science arguments, which describe the destabilizing effect of such a system, add to the complexity of the issue. This multidisciplinary analysis highlights how informed mathematical and scientific thinking can aid citizens in their response to legislative priorities.

noting the 200 billion dollar cost of the renovation. A Boston Globe Science article cited the 1.25 billion heart surgeries performed annually. The absurdity of such numbers is clear if one has a basis for comparison (are these numbers too high or too low and how can one know?).

Skill in estimating real world quantities is enhanced by practice and by knowledge of reference values that provide a scaffold for one's estimates. Students should be familiar with a wide range of benchmark quantities (e.g., populations, distances between points on and off Earth, etc.). Students should create their own number facts sheets for the inside of their notebooks (akin to the integral tables of calculus texts) and have memorized the most frequently needed values (e.g., state and national populations). These quantities can then provide the basis for estimating, for example, the consequences of universal AIDS testing, the number of fish left on Earth, or whether there are a googol atoms in the universe. Estimation is part of checking the reasonableness of results to problems; it is useful to have an estimated, expected answer before more formal, in-depth computation is begun.

For a class activity[7], students are asked to write down, in five seconds each, their guesses for the height of the Empire State Building (or Mount Everest (29,028 ft) or some other well known tall object) and then the number of individually printed letters in the school library (26, though clever, is not the answer). After determining the highest and lowest guesses in the class for each, they are then given time to try to calculate, in some fashion, answers to these questions to check their estimates.

My favorite solution to the first problem comes from Nick who reasoned:

> Fay Wray : King Kong = 1:10
>
> King Kong : Empire State Building = 1:20
>
> Fay Wray is ~6ft so the building is ~1200ft!

The real height is 1250 feet or 1472 feet with its antenna. The building has 102 stories and eclipsed the nearby Chrysler building as the world's tallest building a year after the Chrysler building had earned that title. What I like about Nick's approach is that it demonstrates the importance of each individual deciding which facts known to them will be helpful in solving a problem. Students usually decompose this approximation into estimates[8] about the number of feet per floor and the number of floors in the building. However, Nick's method was certainly accurate and drew effectively on his knowledge base.

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7 The idea for this lesson comes from Douglas Hofstadter's essay "On Number Numbness."

8 I am following the definitions from Lucien Hall's 1984 *Mathematics Teacher* Article "Estimation and Approximation - Not Synonyms." Hall distinguishes between estimations, which are educated guesses, and approximations, which typically require some action such as measurement or calculation in an attempt to reach an answer of a desired degree of accuracy.

The library problem generates an extraordinary range of guesses. These guesses reflect students' unfamiliarity with large quantities. A typical calculation would be: 20000 books in the school library × 200 pages per book × 300 words per page × 5 letters per word = 6 billion letters. In contrast, the New York City public library system has more than a trillion letters or about one for every 5 dollars of national debt (it was 1-to-1 when I started using this question in 1983). As students present their methods for linking a library to its individual letters, emphasize the benefit of using units (e.g., pages/book), writing them out as fractions, and checking that they cancel correctly. The blackboard might look like this:

$1\ \cancel{\text{library}} \cdot \frac{20000\ \cancel{\text{books}}}{\cancel{\text{library}}} \cdot \frac{200\ \cancel{\text{pages}}}{\cancel{\text{book}}} \cdot \frac{300\ \cancel{\text{words}}}{\cancel{\text{page}}} \cdot \frac{5\ \text{letters}}{\cancel{\text{word}}} = 6{,}000{,}000{,}000\ \text{letters}$. We know to write 300 words per page rather than its reciprocal because the units cancel out properly. If we wanted to know how many books we could write with 8 million letters, we would write the ratio the other way around.

Although the above approximation is a combination of several estimations, it can also be seen as a basic unit conversion. One library and six billion printed letters are, in this case, equivalent. This fact can be highlighted by noting that, because 200 pages equal one book (on average), factors such as $\frac{200\ \text{pages}}{\text{book}}$ equal one. Big 1's can be drawn over each factor on the blackboard to show that we are not changing a quantity when we change the units:

$$1\ \text{library} \cdot \underbrace{\frac{20000\ \text{books}}{\text{library}}}_{1} \cdot \underbrace{\frac{200\ \text{pages}}{\text{book}}}_{1} \cdot \underbrace{\frac{300\ \text{words}}{\text{page}}}_{1} \cdot \underbrace{\frac{5\ \text{letters}}{\text{word}}}_{1} = 6{,}000{,}000{,}000\ \text{letters}.$$

I find that students have too often been trained to tackle conversions such as these and the scalings that arise during the solar system model using proportions. Proportions do not efficiently string together as in the above example. Multiplying by conversion factors is easier than proportions to carry out on paper and even easier when using a calculator.

Using conversion factors rather than proportions also facilitates the derivation of conversion factors for two- and three-dimensional units. Students frequently apply linear conversions to higher dimensional measurements and assume, for example, that 12 cubic inches is equivalent to 1 cubic foot. Beginning with 1 cubic foot, we see that we want to remove feet from our unit and introduce inches. Clearly $1\,ft^3 \cdot \frac{12in}{ft} = 12in \cdot ft^2$ does not work[9]. But, it does get us part of the way and continuing the process guided by the exponents, $12in \cdot ft^2 \cdot \frac{12in}{ft} \cdot \frac{12in}{ft} = 1728\,in^3$,

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9 You might want to ask the class what interpretations it can offer for 1 ft^3 = 12 in·ft^2. Twelve inch-high square foot pancake layers should help stress the connection between the units and the physical dimensions.

finishes the job. Handout **N2.2** provides basic information and practice exercises on unit conversion. Students should make these sheets and the solar system data handout the start of a reference section in their notebooks. Students should work on some of the practice problems to be sure that they are ready for the poster and ladder projects which follow and which require many unit conversions as part of these larger challenges.[10]

Students read Douglas Hofstadter's essay "On Number Numbness" (**N3.1**) for homework. This essay discusses the process of developing, and importance of having, an intuitive feel for different quantities in a variety of realms. Hofstadter attempts to persuade the reader of the social and political danger of being numb to the size of numbers and discusses how to approximate unknown quantities using more familiar benchmark facts. Students are asked to choose a few sample problems, called Fermi Problems by many writers, from the last page of the essay and to answer the following:

> Why did Hofstadter write the essay? What were his main points and fundamental concerns?
>
> What is an order of magnitude?
>
> What does Hofstadter mean by 'levels of perceptual reality'? (Give examples).

In class the next day, we discuss the essay and students' responses to it (usually positive with gripes about over-repetitive writing). We discuss Hofstadter's rule that a good estimate should be within 10% of the correct answer at the relevant level of perceptual reality. At the first level, the actual number of elements is perceived. At the second level, the number of digits in the number is perceived. Number names (million, billion, ..., googol (10^{100})) and scientific notation are reviewed as necessary.[11] Students present their approaches to the estimation problems that they choose.

<u>Accuracy and Precision</u>

One of Hofstadter's problems is "How fast does hair grow in miles per hour?" Frequently, a student will express discomfort pulling starting estimates out of thin air (although they are really based on prior experience) and wonder how correct her answer is. In contrast, another student may start with an estimate of 1 inch of hair growth per month and, thanks to her calculator, end up reporting a converted speed of $2.19205948373 \cdot 10^{-8}$ miles per hour. These responses raise the important distinction between accuracy and precision.

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10 A Dictionary of Units of Measurement can be found at http://www.unc.edu/~rowlett/units. This site contains a great deal of useful information. It also lists such fun and unfamiliar prefixes as the yotta- and the zepto- (remember, if you gotta yottameter, you've got a lotta meters (10^{24}, in fact)).

Accuracy is a statement of how close a measurement or estimate comes to the correct value. Precision is a measure of how close repeated measurements come to each other or of the number of significant digits (those in which we have confidence) in a value. For example, if a sonar ranger can only measure a length with precision to one tenth of a meter but has a display that shows hundredths of a meter, then we would expect repeated measurements to reveal variability within the limits of precision. Figure 1, below, can help to clarify these terms. The first student above is concerned because she knows that her answers are not precise and may not be accurate. The second student is reporting far more precision than she has cause to claim. She began with a single digit of precision in their estimate of monthly growth. The varying growth rates of hair would make it difficult to improve upon this value. The calculator flaws revealed by handout N2.1 should be sufficient to begin the process of creating a more cautious reliance on technology. Most of us, however, are easily impressed by the speed and precision of machines. Students should be encouraged to copy from their calculator screen only as many digits of precision as they began with. Hofstadter's fancy expression "level of perceptual reality" is really just a guideline for how precise an approximation should be at different scales.

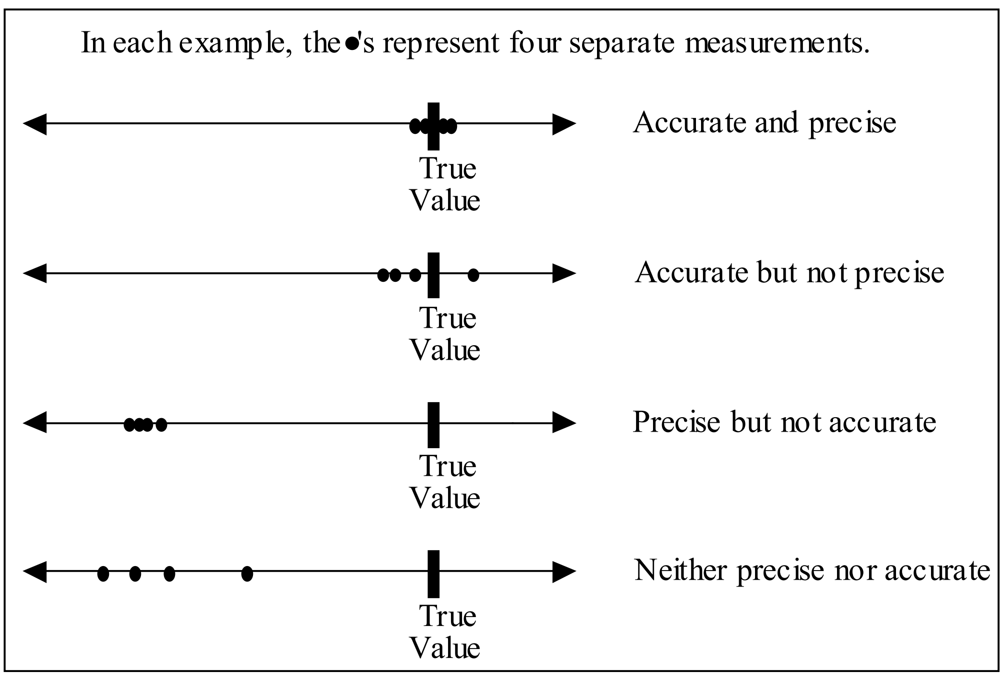

Figure 1. Accuracy and Precision

[11] Note that these names do not have the same meanings in the United States as they do in Europe. For example, a billion equals 10^{12} and a trillion equals 10^{18} in Europe while they equal 10^9 and 10^{12} in the U.S.

Definitions, Simplifying Assumptions, and Research

In class, students try to approximate an answer to "How many trees are on our planet?"[12]

After groups discuss this problem for a few minutes, it becomes clear that the question is not well defined. Students note that there is not a standard for what counts as a tree. Groups heatedly debate the legitimacy of saplings, shrubs, and even seeds (do genes determine tree-ness?). Students need not agree, but they do need to make a choice for their own work (and later they can check if groups counting saplings have a higher total than those that did not). This activity underscores the importance of modelers defining the terms that they use. A related modeling task is the statement of any simplifying assumptions being imposed upon a challenge. For example, one can assume a uniform distribution of trees over the Earth or one can distinguish between forested and tree-less regions. Whichever method students adopt, they should record what each initial estimate represents so that the final result can be put in the context of these simplifying assumptions.

Once "tree" has been defined, students face a variety of geography and geometry questions. In general, they need to ask and answer the following questions:

- What information do we need to know?
- How do we plan to apply it to derive an approximation?
- Where can we find this information?

Depending on their approach, they need to research information on the surface area of a sphere, the radius of the Earth, or the area of the continents. These particular facts are most readily available in almanacs and atlases (which should be available in the classroom). Many values, such as trees per square mile, will not be easily found. Students should be asked how they might successively improve on their estimates of this figure. Often, students need to learn more about a modeling topic before they can pose a meaningful question. The three questions above should be highlighted as part of the initial stages of the modeling process (box 1 in **P1.1**).

Students frequently ask, "but what is the correct answer?" and, of course, I do not know. One can refine one's estimates, but these problems are unknowable as exact quantities. They all have

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12 An additional problem which requires the combination of geographical information, area unit conversions, and estimates is:

> A radio advertisement for Volvo claimed that 17 trillion snowflakes fall in New England each winter. How many fall in your region of the country? (Students in warm weather states may want to approximate raindrops.)

A student of mine was written up in the Worcester Telegram and Gazette for debunking this claim with his own approximation of a quintillion flakes. It turned out that Volvo's advertising agency was stumped in their attempt to find a value, so they made one up.

an unavoidable margin of error. Students learn that precision may be unattainable -- in which case it should be avoided -- but that accuracy is achievable within bounds dictated by the information used. They are also reminded that math problems may be solved in more than one way yielding more than one right answer and that their teacher is not the fount of all wisdom. Why is that last part important? For one, it removes the pressure of having to know everything and allows teachers to experiment, to let classes explore issues for which we may not know the answer. Just as importantly, it puts pressure on the students to check results because there is no expert to fall back on, no back-of-the-book answer key. It also sends the message that, with sufficient effort, they can be the authorities on a subject.

A problem based on arguable estimates and with no single answer strikes students as a risky endeavor. This effect is important because modeling (and learning in general) requires our students to be intellectual risk-takers. The open-endedness of these tasks gives them practice working without a safety net and they gain confidence in their own abilities and judgment along the way.

Exponential Ladders

To be a good estimator, familiarity with, and access to, a range of facts (land and ocean areas, costs, times, etc.) is essential. Lists of exponentially increasing examples, called exponential ladders, are the primary vehicles for developing this familiarity. In class, students work individually on the ladder of mass handout (**N3.2**) and then compare their answers to the answer sheet (**N3.3**). I suggest that answers at the extreme ends that are within two rungs of their correct spot on the ladder are reasonably placed. Because of the variability of some of the items, I have never had a student get the ladder perfectly correct. Exponential ladders are similar in spirit to the book and video Powers of Ten in which each new object considered is ten times larger or smaller than the previous one. For homework, students respond to follow-up questions to the ladder of mass (**N3.4**) and do more of Hofstadter's practice problems chosen by the class.

The Comparison Poster

Memorized information can be quite useful. Ready recall of a fact can facilitate the making of connections that might not otherwise be made. Memorization for memorization's sake (e.g., using flashcards), however, seems to have little impact on our long-term memory. Engagement in rich tasks that require information and are memorable themselves improve the long-term acquisition of information. For the first of two such tasks, the students are given a week to create an original poster to help the class learn key benchmark quantities. Students pick a small set of related quantities (e.g., areas of continents, current national populations, masses of planets, government expenditures, number of individuals belonging to different species) and produce an informative, aesthetically pleasing poster that conveys the quantities and their relative sizes.

As often as possible with major assignments, I show students examples of excellent work done by students in my courses in prior years. With projects that allow the students to choose their topics, it is possible to share an exemplary project without diminishing the challenge of the assignment. In fact, being able to study an excellent effort ahead of time enables the students to analyze the qualities of superior work and helps them produce products that are both different and impressive.

For this assignment, I display several posters that have warranted preservation. Students walk around and study each one. The class then discusses how the graphics and information reinforce each other and how the techniques that were used make the information memorable. Posters must have accurate comparative data of interest and be visually interesting, eye-catching and informative. The graphics should enhance the message and illustrate the relationship somehow. The poster should be sturdy (students typically use poster board). I type up these student-generated criteria as a rubric which I give them the next day (**N3.5**). The class brainstorms a range of possible topics and categories and students pick ideas to research. Others decide to think up new ideas at home. Students must research their facts and put their reference information on the back of the poster (footnoting is always appropriate). Two days into the project, students submit their topic, data, and sources. This submission serves to structure the week and enables me to keep students from spending hours on a beautiful but factually flawed creation.

<u>Teaching tips for the poster project</u>

Poster topics should be about extreme or unfamiliar information that would benefit from a graphic treatment. For example, a poster that compares car prices may not be presenting new information. One that presents different national literacy rates ranging from 50% to 99% may involve important information, but the percents themselves may not benefit from graphic display because they are not difficult to understand numerically.

Note that the rubric is not a checklist that must be fully satisfied. I have received posters that were eye-catching and effective without being attractive (e.g., ones depicting world armaments or deaths in major wars). I have also received beautiful posters that failed to help the viewer understand the extreme differences between the displayed values. Students should try to think of ways to present the relationships between the quantities with more than just a colorful bar chart or graph, although these forms can be effectively used in imaginative ways. They should understand that the project requires an original graphic and that simply enlarging some image they may have found is unacceptable copying in the same way that a paper that contains no original analysis is academically unacceptable.

Often students choose to make graphs that use two- or three-dimensional objects to depict the different facts, but they only use the heights of the objects to represent the sizes of the numbers. For example, they may use people scaled to show populations and make the person for China four times higher than the U.S. figure because China has four times the population. Unfortunately, they scale the China symbol's width as well as height and the area, which is what the viewer perceives, makes China's population appear sixteen times as great. Rather than abandon this format, students should be encouraged to make area the relevant unit and set one square inch or square centimeter equal to a certain number of people. Similarly, a student who built scaled versions of well-known hotels from different cities to represent the populations of those cities needed to use the volume rather than the height of the hotels as the relevant measure (a city with double the population of another had a hotel with eight times the volume, which seemed to dwarf the smaller hotel).

Sometimes students fit a range of lengths onto a poster by showing their logarithm (creating an exponential image in which each inch represents an additional power of ten). One creative student scaled the exponents of their data turning every power of ten into a factor of two when drawn (so an object 100 times longer than another looked four times as long on the poster).[13]

The Exponential Ladder Project

Activity: *The Exponential Ladder Project*

Technical Skills: Converting units.

Modeling Skills: Researching information; working effectively in groups; testing results for consistency.

Audience: Elementary school onward. Older and more sophisticated students should be offered (and choose) more challenging units. Best as a two-person group project.

Materials: Reference sources (preferably a library, but a variety of almanacs, encyclopedia, and texts from biology, chemistry, physics, and astronomy can substitute).

Length: A week with class and home time to work together.

When I think about the abilities that I want students to develop, I look for interesting tasks that will provide practice with one or two particular skills. I then seek challenges that require the combining of many skills at once so that students have constant practice figuring out which they need to apply at a given point in a problem. After the poster assignment, students create their own exponential ladders. Both assignments are sufficiently complex that they give students experience structuring multiple step challenges. The exponential ladder asks students to research examples for each power of ten of a given unit from one extreme of their chosen measure to the other. It expands students' database of example quantities and reveals difficulties they may have

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[13] This data transformation is equivalent to taking the $\log_2 10$*th* root of each number (raising each value to the power 0.301029…).

in using scientific notation, understanding exponent rules, converting units, or recognizing unbelievable results.

Length and mass are good first ladders that get students immersed in almanacs, encyclopedia, and other resources. The data that they find must then be converted to a single common unit (e.g. meters). The project helps students appreciate the significance of matching a number with its unit. They see that a ladder makes no sense without common units yet the units encountered during research for ladders of length include angstroms, centimeters, feet, fathoms, kilometers, miles, and light years. Students learn which units are appropriate in different settings.

The ladder of mass handout provides an example of an exponential ladder, but it does not give students a sense of the full process of the construction of one. Prior to the assignment, the class can do a practice ladder together. Older students might be asked to extend the mass ladder. They could be asked what the lightest and heaviest examples would be and where they would look for values for these masses. They should be prompted to identify the disciplines that would be most likely to provide examples for different ranges not yet covered. This step makes the research more focused and productive. Younger students might begin with the creation of sets of objects and/or populations:

rung	**sets**	**population**
10^0	a pencil	me
10^1	a dozen eggs	our class (24)
10^2	pages in a newspaper	our grade (101)
10^3	times I blink in an hour	our school (527)
10^4	cans and bottles of soda in a supermarket	our town (22,000)
10^5	high school math teachers in the U.S.	New Haven, CT (130,000)
10^6	number of large grains in a cup of sugar[14]	San Jose, CA (780,000)
...		
10^{13}	number of cells in a human being	what organisms might go here?
10^{14}	number of microbes living on each human being[15]	

Creativity plays an important role in making the ladders enlightening and enjoyable to read. The greater the variety of examples, the greater the number of connections readers will make

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14 Based on my own kitchen counting. I took a quarter teaspoon of sugar, divided it in half seven times and counted 40 large grains in the 1/128 of a quarter teaspoon that remained. There are 250 quarter teaspoons in a cup.

between different realms. When students offer multiple examples per rung, the number becomes that much more tangible and the comparisons can be amazing and sobering. For example, the annual per capita income in some countries is equal to the cost of a modest television set in the United States.

Students are paired and given ten days to research and produce their own exponential ladder. They choose one unit (of area, length, volume, speed, time, density, cost[16], or energy) to study. These measures are not all of equal difficulty. For example, a ladder of volume should be roughly 50% longer than one of area and so I counsel students to make their ladder increase by powers of 100 instead of 10. The density ladder is challenging because most values need to be calculated by the student from information about size and mass (e.g., how would they determine the density of the solar system or various atoms?).

The ladders should cover as much of the spectrum from the subatomic realm to galactic measures or beyond. Each item should be placed on the power of ten rung that is closest to the true value of the example. The question of where the dividing line between two rungs is located generates interesting debate. The first suggestion is usually to place measurements greater than 5×10^n on the 10^{n+1} rung and those below that on the 10^n rung. After investigation, students realize that this is not the arithmetic or geometric midpoint. 50 is not equally spaced between 10 and 100 either by addition or multiplication. 55 is the linear midpoint, but the ladder grows exponentially (multiplying by 10 at each step), so the natural dividing point should be an equal multiple away from 10 and 100[17]. This value can be found algebraically ($10x = m$ and $mx = 100$) or symbolically (halfway between 10^1 and 10^2 is $10^{1.5}$) to be 31.4 or $10\sqrt{10}$.

Students need to consult a wide range of disciplinary sources and be adept at converting between various units to succeed in this task. To avoid having students over-rely on a single book or Internet site (and to avoid the plagiarism that that would represent), no more than 30% of the ladder may come from any one source (other than their own knowledge) and all examples must

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15 The remarkable comparison provided by the 10^{13} and 10^{14} rungs comes from Chet Raymo ("Our Bodies, Our Microbes", Boston Globe, July 1, 1996).

16 As with many of these topics, students need to be pushed to figure out examples for the negative exponents. Such examples do exist. The cost of a jellybean, a grain of rice, a single atom of copper in a penny, or a particle of flour suggest possible ways to fill in the low end of the ladder. The unit for cost could be the dollar, but it could also be a rate such as dollars per pound. This latter unit would provide surprises with some cars being cheaper than some candy bars.

17 Asking what makes 1000 "halfway" between 100 and 10000 may help uncover useful confusions or understandings. Similarly, students should be asked to justify how 55 (if it comes up) is halfway between 1 and 1000.

be footnoted. Groups also need to compare their claims for reasonableness. I once received a ladder listing the tallest mountain on the earth as higher than the distance from the earth to the moon. Another correctly listed the diameter of the Sun on the 10^9-meter rung but placed "a sequoia tree in western California" on the 10^{10}-meter step! The images that these facts conjure up are amusing, but their presence in the ladders was not. Such incongruous entries result from errors made converting non-metric researched information into the metric units appropriate for lists that jump by powers of ten. Ladders are successful when they are done in pairs, checked for accuracy by both students, and presented in an integrated fashion. Ladders made from two pasted-together shreds ("you do the high end and I'll do the low end") are rarely as thorough or accurate.

During the ten days of this assignment, students are given ten to fifteen minutes at the start of each class to confer and are expected to meet or talk on the phone as well. Depending on the age of the students and location of the school, it might be appropriate to provide a class period for joint library research during which you would be available for consultations and observation of group dynamics.

Two days into the assignment, each pair has to submit a file card with their names, choice of unit (e.g., grams, meters per second, etc.), and the smallest and largest examples thus far researched (e.g., the mass of a cell and the mass of the Sun). The importance of sticking with one unit is repeatedly emphasized. It does not make sense to list a soccer field on the 100-meter rung above the length of Manhattan on a 10-kilometer rung. It also makes no sense to compare different types of units such as a temperature of 1,000,000 degrees Kelvin versus a density of 1000 grams per cubic meter. Students should be reminded to avoid reporting too much precision while being sure to provide full descriptions (e.g., in a ladder of length, the entry "Sun" could mean its radius, diameter, circumference, or distance to the Earth). Lastly, they need to make certain that they and their readers understand what the examples really mean. I have received items that, for all they meant to me, read like "10^{12} seconds - the time it takes for a snork to galumph through its pharnex." It is great if kids find technical or unfamiliar information, but they must then do the research that enables them to understand and explain their finding.

After submitting their ladders, the class is shown the video *Powers of Ten*, which provides a striking visual example of an exponential ladder of length. I evaluate ladders according to the criteria discussed above and their length. I am not concerned if the ladders have gaps for extreme rungs. Both mass and volume ladders become tedious at the extremes because a ten-fold increase in volume only corresponds to a $10^{1/3}$ or a 2.2-fold increase in length. However, the gaps should not appear in our everyday realm (e.g., no example for 10 meters) and they should still be listed in the ladder so that items remain consistently spaced according to their size. Document **N3.6** provides examples of successful student ladders (which may, or may not, exhibit all of the qualities which I have noted).

Number Systems and Properties

Number magnitude is the main theme above. The following activities emphasize the importance of understanding integer, rational, and decimal representations of numbers. One byproduct of the increased use of technology has been a de-emphasis of rational number systems. This diminished attention is not necessarily good nor is it an unavoidable byproduct of calculator use. Although fraction operations are often built into calculators for younger students, they are usually omitted from calculators designed for high school courses. This omission leads to a move away from exact answers even when they are simpler and more informative. When students have decided to study a problem numerically, they need to make the additional choice of which number representation is likely to be most helpful. Many surprises arise from a study of different number types, their representations, and their properties.

Another theme that emerges in this unit is the value of rigorous mathematical reasoning and the role which algebra plays in turning specific observations into conjectures and theorems. Students are too prone to believe that a few examples constitute irrefutable evidence of a pattern. Often these patterns only exist, if at all, for certain number classes. Conjectures must be tested for varying cases (e.g., $x^2 \geq x$ when x is an integer but not for all real values of x).

During class, students name all of the number types that they know. Depending on the background of the students, the list may contain many of the following: counting, whole, integer, rational, irrational, transcendental[18], real, positive, negative, prime, composite, prefect square, odd, even, imaginary, and complex[19]. Students individually write definitions for each type listed. As the list grows on the board, use columns to distinguish between number types and representations. For example, students may define rational numbers as those that can be written as a fraction (*ratio*-nal) or as a terminating or repeating decimal. These are two representations of the same number type. Have them identify which types of numbers are subsets of others and which overlap. Making a Venn diagram of these sets can be particularly challenging. I ask for examples from each category and how they could be located or constructed on a real number line (see figure below). This question reinforces the definitions and relationships between the values. Students should create their own reference page on number types, definitions, and properties.

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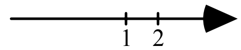
18 Transcendental numbers are a subset of the irrationals. Irrationals are either algebraic (the solution to a polynomial equation with rational coefficients) such as $\sqrt{5}$ or $2 - \sqrt[7]{6}$ or transcendental such as π, e, sin 1^R, or non-repeating decimals such as 0.101001000100001000001....

19 Handout **N4.0** is a useful addition to students' reference sections.

Given this number line, 

, how would you locate 0, -4, $\frac{3}{7}$, $\sqrt{5}$?

Homework includes the reading *Zero* by Constance Reid (**N4.1**) on the history of the symbol '0' and the meaning of expressions such as $0/c$, 0/0, and $c/0$. It is important to note that the numbers with which we are so comfortable have all been invented (or discovered depending on your perspective) over time. As new problems arise, new numbers have been employed to solve them. Each of the following equations requires a new type of number in order to have a solution: $x + 4 = 7$, $x + 7 = 4$, $4x = 7$, $x^2 = 7$ (x is an algebraic irrational), $\cos x = 0$ (x is transcendental), $x^2 = -4$, $x^2 + x + 1 = 0$. It is also crucial to emphasize that the names given these numbers can be misleading. Real numbers are not any more real than imaginary ones when one considers the substantial number of applications for both.

Class discussion of the *Zero* reading usually reveals continued confusion regarding the distinctions between zero ($0/c$), indeterminate (0/0), and undefined ($c/0$) expressions. After students offer their interpretations, I explain the expressions using the concrete, "goes in to" view of division that students first learn. If 8/2 asks how many times the denominator 2 goes in to 8, one can imagine an 8 gallon vat and ask how many times you would have to pour a 2 gallon container in to it in order to fill the vat. Similarly, 0/2 asks how many times you need to pour in a 2-gallon jug to fill a vat that holds no liquid. The answer is, it is already filled and any further pouring would exceed its limits. Therefore 0/2 = 0. An answer for 2/0 could be infinite because you can pour 0 into a 2-gallon vat forever without finishing the task. Sometimes, depending on the context in which the expression arises, it makes sense to call 2/0 undefined and sometimes it is helpful to recognize that something infinite has happened (e.g., $y = 1/x^2$ is undefined at $x = 0$, but the values, and graph, of the function are growing without bound as x nears zero). Lastly, 0/0 could equal 0 because the vat requires no filling. But, it could also equal 6 because after pouring 0 into the vat six times it is still exactly filled with the nothing it requires. These distinctions are both basic and sophisticated. An understanding of most functions (especially those with asymptotic behaviors) and of Calculus requires that they be mastered[20].

Understanding Definitions: *Closed* and *Dense*

The value in identifying categories of numbers comes from the fact that the different sets behave differently. As the preceding sequence of equations demonstrated, certain operations require certain types of numbers to permit their general use (e.g., taking a square root is rather limited if only integer results are allowed). The properties of each set can be understood through

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[20] The Texas Instruments TI-92 has a symbolic manipulation system built in that gives the following answers to the following entries: $\frac{1}{0}$ is undefined, $\frac{1}{0^2}$ is $+\infty$, and $\frac{1}{-|0|}$ is $-\infty$. Can you explain these differences (consider the graphs of $\frac{1}{x}$, $\frac{1}{x^2}$, and $\frac{1}{-|x|}$)?

an analysis of behaviors that are clearly defined and testable. Definitions of properties and objects play an important role throughout pure and applied mathematics.

Until an idea is formally stated in a definition, it is often not recognized or a subject of study. For example, *perfect* numbers, whole numbers whose factors sum up to twice the number itself (6 is perfect because $1+2+3+6 = 2 \cdot 6$), have been the source of many interesting investigations in number theory, but they had to be identified and defined before they could be explored. The development of mathematical models frequently requires the creation of definitions that help to translate real world considerations into mathematical terms (see More Ranking Functions in the Functions chapter). What is meant when a company wants the most *efficient* manufacturing process? Efficiency can be defined and then optimized in terms of energy, materials, labor, time, money, or a combination of all of these. The writing and interpretation of clear and useful definitions is an essential mathematical skill[21].

Students are introduced to two new definitions in order to aid them in distinguishing between the number types. The first definition is: A set is *closed* under an operation if applying that operation to members of the set always results in members of the set. The students read the definition and discuss what steps they need to take to gain a complete understanding of it:

1) Read the definition more than once.
2) Identify what "things" the definition is talking about. They note that *closed* is talking about a set and an operation that acts on elements of that set.
3) Generate a test case. The words "set" and "operation" were really place holders, so having picked an example of each, the integers and addition, the students re-read the definition substituting in their choices: The integers are closed under addition if the addition of integers always results in integers. For a geometry definition, draw a diagram that matches the defined situation.
4) Determine if the example fits the definition. Does adding integers produce integers? This step requires consideration of different types of integers, such as positives and negatives, to see if integers must result. It is important to note that the definition requires all cases to work and that there are an infinite number of possible additions. Clearly, a claim of closure must be based on a sound reason rather than a few experiments.
5) Find examples that do not fit the definition. The identification of non-examples strengthens one's understanding of a definition and highlights the importance of each condition of the definition. Non-examples can be constructed systematically by removing each condition (or key word) of the definition in turn and finding an example that satisfies

[21] For several activities to use when teaching about reading and writing definitions, see Definitions at the Making Mathematics web site (www2.edc.org/makingmath/handbook/definitions/definitions.asp).

all of the remaining conditions but not the excluded one. For example, the reals are not closed under division despite the infinite number of divisions that do produce a real result. A single counter-example such as 8÷0 suffices to answer the question and emphasizes the role that "always" plays in the definition.

6) Try to restate or explain the definition in your own words.

As the above outline demonstrates, reading mathematics is an active process. It should be done with pen and paper available and should be regularly interrupted by the reader's attempts to test, apply, and develop their understanding of an idea.

Closure is a valuable idea to explore because the failure of closure is what has repeatedly led to the introduction of new number types. For example, the whole numbers are not closed under subtraction, so negatives must be introduced. The reals are not closed under taking square roots, so imaginaries must be introduced[22]. As with some other activities and topics that will be discussed, closure is presented without any attempt to directly connect the idea to applications of mathematics. It is a tool for emphasizing habits, the reading of definitions and testing of cases, that are essential to modeling work. It is those habits that are connected to the modeling cycle.

In order to show that a set is not closed, students must find a counterexample, an instance that contradicts the definition. Finding a counterexample to a claim requires a thoughtful search. Test cases must be generated which cover the range of different outcomes. What happens if the operation is applied to two of the same value? What if one value is much bigger than the other? What if they come from different subsets of the set in questions (e.g., one is prime and one is composite)? Any mathematical claim invites different test challenges. Students should be encouraged to be sure that their tests of different circumstances include extreme cases. What happens when the smallest (or closest to zero) and biggest values are checked? As with the 10^{50} - 10^{20} problem, surprises appear when we actively change directions and look for them.

Students explore the definition of *closed* in class with the following problem:

$S = \{1,2,4,5,7,8,10,11,13,14,16,17...\}$.

How would you describe the members of *S*?

Is *S* closed under:

a) addition? b) subtraction? c) multiplication?

d) division? e) exponentiation?

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[22] Interestingly, complex numbers are closed under radical operations. The square root of a complex number is just another complex number and no new number system is required. To find the square root of a complex number, set it equal to the general complex number $a + b$i. For example, to find the square root of i let $\sqrt{i} = a + bi$ and solve (by squaring) for a and b.

(In each case say “open” and give a counterexample or “closed” and write a sentence or two that prove that no counterexample could exist.)

S, the set of all whole numbers without three as a factor, is not closed under division: 5/7 is not even a whole number. *S* is closed under multiplication because the whole numbers are and because the product of two non-multiples of three cannot have three as a factor. A homework assignment (**N4.3**) provides further practice.

Counterexamples may be common (half of the additions in *S* yield a multiple of three), but they are frequently atypical cases. A single counterexample will suffice to prove the lack of closure, however the mere testing of cases can never prove a set is closed under some operation (unless a claim involves a finite set). For example, on the take-home numeracy quest (**N5.1**), students test the set of reals between 0 and 100 for closure under taking the reciprocal. Any whole number tested does indeed land back within the set (e.g., the reciprocal of 12 is 1/12 which is still between 0 and 100), but numbers less than 1/100 have reciprocals that are too large and so the set is not closed under this operation.

The above question is a good test for *integeritis*. Integeritis - the dreaded inflammation of the integers - is a common disease about which I warn my students. It involves a general failure to think about fractional quantities and can lead to many mistaken assumptions. Integeritis in its larger sense is the arbitrary choosing of values rather than the informed exploration of a process. The problem on the take-home quest will defeat a random search for counterexamples. The solver must try to identify values that will land outside of the given range and integers simply will not do the trick. To help cure students of integeritis, I assign a handout that emphasizes the different behaviors of numbers greater than and less than one (**N4.4**)[23]. One extension of this exercise suggested by students is to test Small and Large numbers for closure under division. Doing so reveals that the ratio of two Small or two Large numbers can be close to zero or as large as desired. These observations nicely pave the way for an understanding of why limits that evaluate to 0/0 or ∞/∞ are indeterminate.

Another tool for distinguishing between the different number types, and between sets of numbers in general, is the property of *dense*-ness of a set. The class investigates this definition: A set is *dense* if between any two distinct members of the given set, there exists a third distinct element from the set. One consequence of this definition is that there must also be an infinite number of elements between any two elements because the definition can be applied recursively to an original element and the new element between the two. Can the students identify which of their number sets are dense and which are not? Can they argue their case? The reals are dense

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23 The definitions on the handout are intentionally difficult, but still accurate, to get students to practice reading and testing definitions.

which tells us that any geometric segment is dense. Possible questions are: What is the number after 11? The answer depends on the numbers under consideration. If integers are allowed, the answer is 12. If reals are permitted, there is no answer[24]. Are the rationals dense? If so, can they show that there is always another rational number between a/b and c/d?

Proof and Algebra

Once a mathematical model is created, new knowledge will only arise from the thoughtful and creative manipulation of the mathematical objects that constitute the model. There is no absolute distinction between pure and applied mathematics — investigations in each realm inspire and support advances in the other. Modelers must be proficient developing logical and well-understood mathematical systems if they want their models to be trustworthy. An undiscovered counterexample to a claim underpinning a model might point to real world behaviors that needed to be anticipated. It is appropriate, therefore, for mathematical modelers to spend time on abstract explorations and proof.

Proofs generally require the demonstration that an infinite number of cases all display a common property. All right triangles satisfy $c^2 = a^2 + b^2$. All whole numbers have a unique factorization (the fundamental theorem of arithmetic). How can students show that every possible pair of elements chosen from a set satisfy the definitions of *closed* or *dense*? English arguments can be effective. For example, a student might claim that even integers are closed under multiplication because each element has a factor of two and therefore the product must also and thus meets the definition of evenness. Formal notation is unlikely to make such a statement clearer. Pictures can be convincing. *The College Mathematics Journal's* "Proof Without Words", now more timidly called "Mathematics Without Words", regularly provided persuasive, if not technically rigorous, diagrams.

The most general method of proof available to high school students is algebra. Students use variables to represent both numbers and geometric objects when the variables are presented in textbook exercises. However, students are rarely comfortable introducing variables into settings themselves. The common instinct is that a variable is an *unknown* and they do not see the value of adding more uncertainty into the problem-solving situation. They are less likely to see variables as *variables*, as symbols that simultaneously hold the place for all values of a set. Helping them come to see the utility of naming quantities with variables is worth considerable effort.[25] A focus on using variables within abstract proofs can transfer to students' work creating models.

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24 The reals can be identified with all of the points on a line. The geometric consequence of no two numbers being neighbors is that no two points on a line are adjacent. Students can stretch their intuitions by trying to reconcile

Proving that the evens are closed under multiplication is a straightforward task, but students have a more difficult time proving that the odds are also closed under multiplication. Student claims that an odd times an odd always equals an odd are simply a statement of past observations. As a reminder of our inclination to trust prior observations, the slogan "an example is not a proof" ought to be posted prominently in all classrooms. Given a variable, n, which represents all integers, the class can be asked how to make an expression which is always even and then, from $2n$, one that is always odd. The product of two odds can then be translated into $(2m+1)(2n+1)$[26], with n and m restricted to integers, which distributes to $4mn+2m+2n+1$. What does the class make of this expression? Can they show whether it is even, odd, or varying? If it can be put into the form $2a$ where a is an expression that is always an integer, then it is even. If it can be put into the form $2a+1$ where a is an expression that is always an integer, then it is odd.[27]

The following two proofs that the rationals are dense have been created in my classes. The first finds the average of any two given fractions and the second adds the numerators and denominators. The latter method makes a good challenge for students to explain. Why, in terms of the meaning of ratios, is the resultant fraction always between the initial two?

Given: any two fractions, $\frac{a}{b}$ and $\frac{c}{d}$ with integers a and c, whole numbers b and d, and $\frac{a}{b} < \frac{c}{d}$.

Show: there is a third rational number between the given two.

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the continuity and denseness of lines.

25 The best approach is to choose algebra curricula that regularly require students to create their own expressions to represent quantities. The program *Algebraic Patterns* (available for $15 from Visual Mathematics at www.visual-math.com) is a fine tool for providing such practice in a setting that emphasizes looking for patterns and conjecturing.

26 Note that $(2n+1)(2n+1)$ is not acceptable, because for any particular substitution of a value for n, the expression is always the product of an odd number with itself. Students should also note that m and n can be equal, so $(2m+1)(2n+1)$ does indeed cover all possible pairs of odd numbers.

27 Another practice problem: Are the perfect squares closed under addition? Multiplication? Exponentiation? Give a counter-example or proof to support each claim.

Proof 1: Show that $\frac{a}{b} < \frac{\frac{a}{b}+\frac{c}{d}}{2} < \frac{c}{d}$.

$\frac{a}{b} < \frac{c}{d}$	Given.
$\frac{a}{b} + \frac{a}{b} < \frac{a}{b} + \frac{c}{d}$	Added the same term to both sides in order to get an expression that looks similar to the middle term in the above inequality.
$\frac{a}{b} < \frac{\frac{a}{b}+\frac{c}{d}}{2}$	Divided both sides by two. We have not met the conditions of *dense*-ness, until we show that the right hand side of this inequality is indeed a rational number, too. Two lemmas[28] demonstrating that the rationals are closed under addition and division are needed to show that $\frac{\frac{a}{b}+\frac{c}{d}}{2}$ is rational (more good exercises for the class). For these lemmas, it is reasonable to take as a given that the integers are closed under addition and multiplication.
$\frac{\frac{a}{b}+\frac{c}{d}}{2} < \frac{c}{d}$	Adding $\frac{c}{d}$ to the given and dividing by 2 completes the proof.

Proof 2: Show that $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. It is not obvious how to turn either starting fraction into the middle expression. An approach can be found by working backward from the goal:

$\frac{a}{b} < \frac{a+c}{b+d}$	Where we want to end up, so this is not our proof. Let's get rid of the fractions and see what happens.
$a(b+d) < b(a+c)$ $ab+ad < ba+bc$	Multiply by $b(b+d)$. This expression is positive so the direction of the inequality is preserved. Distribute.
$ad < bc$	Subtract ab from both sides. Oh, we've stumbled onto something familiar looking.
$\frac{a}{b} < \frac{c}{d}$	Divide by bd, which is positive, to get our given.

The above sequence began with what we hoped to prove, so it is not a proof itself. We can generate the proof if each of the steps is reversible:

$\frac{a}{b} < \frac{c}{d}$	Our given.
$ad < bc$	Multiply by bd, which is positive.

28 A lemma is a side proof that simplifies the presentation of a theorem.

$ab+ad < ab+bc$	Add *ab* to both sides. Note that this step appears unmotivated if we do not first share the above exploration.
$a(b+d) < b(a+c)$	Factor.
$\frac{a}{b} < \frac{a+c}{b+d}$	Divide by $b(b+d)$. This expression is positive so the direction of the inequality is preserved. The integers are closed under addition and $b+d \neq 0$, because *b* and *d* are both positive, so $\frac{a+c}{b+d}$ is a rational number. A similar sequence proves the upper limit of our inequality. QED[29].

These two examples illustrate many basic strategies that arise in the search for proofs: looking for common symbolic structures, working backwards, and finding helpful symbolic versions of our given information. The challenge of beginning a problem with a useful representation is a crucial one. Students, quite reasonably, may try to make their given the numbers *a* and *b*, both rational. However, use of two variables to represent each fraction captures more of the situation symbolically and makes it easier to find a proof (at least for the second proof).

The distinction between examples and a proof can be made whenever members of the class disagree on a homework or class question about closure. They can be asked why some of them believed that an open set was closed. As they are forced to repeatedly recognize for themselves their tendency to trust examples, they will grow more cautious and careful. A classic activity for reinforcing this point involves $x^2 + x + 41$. Have the class begin a table with the first dozen or more values of the polynomial when $x = 0, 1, 2, 3$, etc. Seek observations and conjectures about the values that arise. All will be both odd and prime. Some may also note the arithmetic sequence of the differences between the terms. As students extend this table, they will see that the patterns continue. Are they convinced that the patterns will continue forever? How many terms are needed to sway them? What if they actively look for values of *x* that might not yield a prime or odd number? The first forty terms will be prime, but the polynomial yields a composite number $(41 \cdot 43)$ when $x = 41$.

There are many reasons for spending time on proof in a modeling course. Making algebra a more appreciated and understood tool was cited above. Studying proof highlights the importance of thinking rigorously and seeking evidence to support one's opinions. Despite what geometry texts would have us believe, proof is not an after-the-fact activity. Often proof develops out of an exploration of examples and an attempt to understand them. Conversely, the process of creating a proof may just as often enrich our understanding of a problem.

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29 An abbreviation for quod erat demonstrandum, which means, "that which was to be proved" or "Ta-da!"

Another reason for including proof in a modeling course is that students have too little experience trying to prove or disprove claims without knowing the outcome ahead of time. These experiences help kids become more questioning, skeptical mathematicians in both pure and applied situations. The sections below on $.\overline{9}$ and Cantor's infinities explore several counter-intuitive ideas. What makes certain ideas counter-intuitive is that our experiences have provided many examples that lead us to believe a certain fact and have not revealed to us the cases that suggest a different truth. I regularly chant the mantra "an example is not a proof" and try to show students the error of leaping to conclusions. Of course, mathematics conjectures often arise from a study of examples and the inductive leap to predicting a pattern. It is important to recognize that our intuitions both guide us to truths and trick us regularly. Proofs and counterexamples ought to be the ultimate test of a claim.

Rational Number Representations and a Surprise

Many of the facts that we learn are taken for granted until closer questioning shakes our faith in what we believe. How, for example, do we know that rational numbers must have repeating decimal representations[30]? Conversely, how do we know repeating decimals must have a rational representation? Assignment **N4.5** describes methods for converting between fraction and decimal representations of numbers and includes practice problems. These problems extend the unsettling experiences begun by division involving zero. They force students to question their thinking about numbers that they thought were well behaved and that they felt they fully understood. Students can read the handout without further introduction. The reading can also supplement a class investigation into the first question above[31] and a presentation of the method for converting repeating decimals into fraction form.[32]

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30 Note that a distinction between repeating and terminating decimals is unnecessary. 0.5 is also 0.500000.... which is repeating.

31 A question about a particular instance, such as "How do we know that fractions such as $^1/_{11}$, $^2/_{11}$, etc. will always repeat?" might be a good parallel question to pose along with the general inquiry. Middle school students, having more recently studied long division, may be less removed from the issues involved here and may respond more comfortably than older students.

32 Many students have weak fraction skills. John Dewey claimed that basic skills were not learned best through practice, but by engaging in rich tasks that required them. Having witnessed high school students in General Math classes (another invented school math subject like Algebra II and Precalculus) unsuccessfully relearn fractions for the fourth time, I am compelled to agree. Fraction skills will receive plenty of practice supported by insight if instead students spend a class or two posing questions, making conjectures, and seeking proofs about fractions. One good starting point is to define *unit fractions* as a fraction with a numerator of 1 and a counting number denominator, to note that $^1/_3 + {}^1/_6 = {}^1/_2$, and to ask what questions are suggested by this instance. The

One of the **N4.5** problems asks students to convert $.\overline{9}$ to a fraction. The steps $N = .\overline{9}$, $10N = 9.\overline{9}$, and $9N = 9$, yield the surprising answer that $N = 1$. This situation never fails to produce a lively debate. Does $.\overline{9}$ really equal 1? Unlike the other cases, which permit you to check your answer by dividing and seeing if the decimal is produced, there is no obvious way to turn 1 back into $.\overline{9}$ through long division[33].

Some students claim that $.\overline{9}$ and 1 are very close in value but different. I ask if they trust the conversion technique in other cases but not in this one. I also ask if they can find a step in the process that they distrust. Some question the step in which the decimal place is shifted. We discuss whether anything sneaky is happening as an infinite number of digits have their place value changed. Students will usually challenge the claim of a difference or produce an expression for that difference. Their reasoning is usually presented as follows:

1.0 - .9 = .1

1.00 - .99 = .01

1.000 - .999 = .001 etc. so

1.0000... - .9999... = .0000... "with a one at the 'end'" = $.\overline{0}1$

At this point, someone will point out that there is no end to the infinitely repeating zeroes so there is no "1". Without a "1", the difference is zero so the numbers are identical. One student once responded that there are an infinite number of numbers between 0 and 1 on the number line so you can have an infinite number of 0's in a finite space between the decimal and the one: 0.00000000001 ! His classmates did not accept this typographical solution!

Another argument that students have made is that if 1 and $.\overline{9}$ were different, then by the denseness of the reals, there would have to be a number between them. Since they could not think of a way to squeeze a decimal value between the two, they must be the same[34]. Others rebut that decimal representations are unique (their assumption is based on experience with *examples*) and so these must be "slightly" different numbers.

Some have pointed out the following patterns:

.3333... = 1/3 so multiply both sides by three to yield:

.9999... = 3/3 = 1

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patterns of fractions which satisfy $^1/_a + ^1/_b = ^1/_c$ are quite interesting and many other interesting generalizations of this situation exist. A memorable experience with surprises and good discussions of the claims that students make is likely to produce sounder fraction work in the future.

33 Although, I realized as I wrote this that it could be forced if you try!

or

1/9 = .1111...

2/9 = .2222...

3/9 = .3333...

....

1 = 9/9 = .9999...

These observations prove particularly disturbing for students who question the equality of the two representations 1 and $.\overline{9}$. Now, accepting 1/3 = $.\overline{3}$ becomes problematic, because it forces acceptance of the other equality. That equalities, which had seemed reasonable, effective, and simple, were now proving to have unwanted consequences shocks students who think that our numbers behave in intuitively obvious ways. I have had students decide that 1/3 is also "just close" to $.\overline{3}$ even though long division of 1 by 3 clearly generates a repetitive process. Throughout the discussion, I attempt to play Devil's Advocate, questioning claims on both sides. When asked which side is right or which arguments are true, I decline to offer an opinion and force them to try to convince each other. Many of these student-generated ideas have arisen because of that unwillingness to play expert and settle the matter.

The cause of the students' discomfort lies in how they think of repeating decimals. One of my students articulated what she and her classmates were feeling. She called $.\overline{9}$ an "active number" because it was actively in the process of becoming 1. She appreciated the difficulty of distinguishing between the process of repeating and the product or *limit* that results from infinite repetition. I do think of $.\overline{9}$ and 1 as equal because I consider repeating decimals not as becoming, but as being there, in their full infinite extent. We do not speak of 1/3 as becoming $.\overline{3}$ nor is the ratio between the circumference and diameter of a circle becoming π, it is irrational and infinite at once. This is how numbers are generally discussed, however, there are different theoretical constructions of the real numbers and it is not necessary, nor fruitful, to be dogmatic with the students. Their own future mathematics experiences will continue to refine their thinking on this question.

Presenting this class discussion in detail highlights two points. Students are not used to having to present reasons for what they do or claim in mathematics classes. Surprises such as this one motivate them to engage in that habit. They discover that an argument that they find convincing (usually because its conclusions coincide with their intuitions) may not sway their peers. They learn that the validation of a proof or line of reasoning is dependent on the consensus of their own mathematical community. Students should be encouraged to respond not only with

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[34] This argument is a perfect proof by contradiction. If it is made, it should be pointed out at the time as such or the next day during the dollar bill activity.

debate points ("I agree because...", "I disagree..."), but also with requests for a classmate to clarify a comment or with questions about the meaning of words or symbols being used. This exploration demonstrates what can occur when teachers pose good questions and then step back and listen. What we start to hear is the subtle differences in the way our students think about concepts that are at the heart of our discipline. The more we listen, the more complex our understanding of the distinctions and connections that our students' are struggling to make becomes.

The goal of this experience is to expose students to the subtlety of something they did not think was subtle and to have them think about infinite processes and objects. It is valuable for students to develop sophisticated intuitions about situations involving infinity well before they are immersed in them on a daily basis in a Calculus course. When students ask why we are studying confusing, paradoxical numbers such as $.\overline{9}$, I respond with these reasons and also extol the fun, surprising, and aesthetically intriguing aspects of the problem. Finally, I note that the real numbers and number lines are our model, or at the heart of models, for much that we study when applying mathematics. Thus, it is important that we understand the extent to which our model is like the real world and what strange properties may be lurking in our mathematical representations.[35]

Comments on and Selected Solutions to Handouts from this Section

N4.3: The most specific number category is listed for each item:

$^{13}/_{8}$ - rational	$\sqrt{35}$ - irrational
$\sqrt{36}$ - counting	$\sqrt{-9}$ - none of those listed
-321 - integer	$\sqrt{15}\cdot\sqrt{15}$ - counting
π - irrational	3.14159 - rational
.010267267267267267… - rational	3.12233344445555566666... - irrational
$-^{12}/_{3}$ - integer	

The counting numbers are open under subtraction (5 - 9 = -4) but closed under multiplication. The integers are closed under multiplication but open under division (3 ÷ 7 is not an integer). The rationals are closed under multiplication ($\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$ and *a*, *b*, *c*, and *d* are integers which are closed under multiplication, so *ac* and *bd* are integers as well making the result rational) but open under division (0 is a rational number and division by zero produces non-real results).

[35] Fun follow-up questions to this activity: If you accept that $.\overline{9}$ = 1, then which other real numbers will have two decimal representations? Does this situation occur in other base systems?

The odd integers are "way open" under subtraction (every case is a counterexample), closed under squaring (expand $(2n + 1)^2$), and open under raising to powers (3^{-3} is not an integer).

The irrationals are open under addition: $-\pi + \pi = 0$, $\sqrt{11} + (1-\sqrt{11}) = 1$, 0.101001000100001000.. + 0.010110111011110111... = 1/9, or 0.184620098312… + 0.815379901687… (with random complementary expansions that sum to 9 in each place) = 1. They are open under multiplication as well ($\sqrt{15} \cdot \sqrt{15} = 15$).

N4.5: After reading the fraction conversion description, students might be asked what they think of the line "This same line of reasoning applies to any division involving two integers." which follows the long division examples. Is it convincing? Is it a proof? Can the students make it more rigorous or clarify and generalize what the line of reasoning referred to is?

Note that the technique[36] used to cancel out the infinitely repeating digits can be used to simplify other series as well. Repeating decimals are simply infinite geometric series (e.g., $0.3333\ldots = {}^{3}/_{10} + {}^{3}/_{100} + {}^{3}/_{1000} + {}^{3}/_{10000} + \ldots$ with starting value ${}^{3}/_{10}$ and common ratio between terms ${}^{1}/_{10}$). Setting the general form for a geometric series $a + ar + ar^2 + ar^3 + ar^4 + \ldots$ equal to N, multiplying both sides by r, and subtracting the two equations, yields $N = {}^{a}/_{(1-r)}$. The formula for finite geometric series is similarly derived.

1c) $0.\overline{0588235294117647}$ 1e) Not rational, so it cannot be written out fully.

2b) ${}^{4}/_{33}$ 2c) ${}^{2}/_{9}$ 2d) 1

3) Long division works, but a student may also use a calculator and look at the pattern generated by 10/11, 100/101, 1000/1001, etc. If this approach arises, note to the class that it is a nice application of the test small cases problem solving habit.

5) A symbolic mathematics program such Mathematica™ can generate decimal expansions of any desired precision. Because the long division algorithm only requires integer calculations, a student might also try to write a computer program to generate and identify the repeating patterns.

Proof by Contradiction and The Existence of Irrationals

Two students come to the front of the room and I show them and the class one \$10 and two \$1 bills[37]. I have them close their eyes. I ask each to raise a hand over their head where they cannot be seen. I place a \$1 bill in their raised hands and put the \$10 bill away. When they open their eyes, each is able to see the other's bill but not their own. Before instructing them to open

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36 A technique is a trick used more than a once.

37 The idea for this activity is taken from the teacher's notes of Harold Jacobs' Geometry published by W. H. Freeman.

their eyes, I explain to them the situation (not the part about which bill they were given) and ask them to tell the class when they know whether they are holding a $1 or $10 bill. Once they open their eyes, there is usually a long period of time where both students grow progressively more awkward or giggly as they stand there unable to tell whether they have the $10 bill or the leftover $1 bill. If they say they know, they are asked for their reasoning. The claim that I would not give them a $10 bill is not deemed a solid proof! Ultimately (this can take a few minutes of squirming and some eventual reassurance that they can and will figure out the problem), one student announces that if they had the $10 bill, their classmate would instantly announce that they have one of the $1 bills. Since the classmate seems confused, then the proof-giver must also have a $1 bill. I write out their reasoning parallel to a list of the steps of an indirect proof[38]:

Your Claim	I have a single.
Assume the opposite of what you are trying to prove.	If I have the ten,
Show that a contradiction is generated by this assumption.	Then they would know they have a single, which they don't seem to know.
Conclude that the assumption, in leading to a contradiction, was false.	Therefore, I cannot have a ten.

With this not so subtly timed background (who says every activity has to be authentic?), the following question is posed: how can we prove that $\sqrt{2}$ is, as identified previously, irrational? How do we know that the decimal representation never repeats?[39] When the class realizes they cannot prove the expansion never repeats, they try a proof by contradiction. The class slowly works it way, usually with help from me, through the following proof:

Assume that the square root of two is rational:

$\sqrt{2} = a/b$, with a and b counting numbers and the fraction in lowest terms.

then $2 = a^2/b^2$

and $2a^2 = b^2$

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38 A proof by contradiction (also called an indirect proof) of the statement "if *a* then *b*" is really a direct proof of the contrapositive statement "if not *b* then not *a*." Since statements and their contrapositive are logically equivalent, either proof suffices. Note that proving the inverse "if not *a* then not *b*" or converse "if *b* then *a*" tells us nothing about the truth of the original claim. See Jacobs' geometry text for a good discussion of these distinctions.

39 This question prompted a student to ask whether we could treat infinity or quantities with infinite extent like normal numbers. She was extending some of the concerns from the $.\overline{9}$ discussion and realizing the pervasiveness of ideas and representations involving limits.

These steps reduce the problem to one involving only counting numbers instead of roots and remove the likelihood that any not-yet-proven assumptions about roots or irrationals will creep into the reasoning. At this point several arguments have been tried. One is that b^2 must be even because the left side has a factor of 2 and therefore *b* must be even[40]. Then it can be shown, by substituting $b = 2n$, that *a* must be even. If both are even, then a/b is not in lowest terms as assumed. This contradiction means the original assumption must be incorrect and $\sqrt{2}$ cannot be represented as the ratio of two integers. One student noted that there must be an odd number of prime factors on the left side (counting duplicate factors separately) and an even number of prime factors on the right side and it is impossible to factor equal numbers into primes differently. This latter explanation proves much more accessible and memorable. Students frequently view the initial condition that *a* and *b* are already reduced as a gimmick they do not understand. The approach involving the parity of the factors has greater appeal even though it relies on two additional claims: the fundamental theorem of arithmetic (numbers have a unique prime factorization) and the claim that perfect squares have an even number of prime factors (can students support this?).

For a nice connection to the study of bases in our computer class, I present a nifty proof from the March 1991 College Mathematics Journal (vol. 22 no. 2), which uses base three. The technique can actually be used in base ten in which the final digits of perfect squares can be used to demonstrate the irrationality of 2,3, etc.[41] All of these proofs are summarized in Martin Gardner's article listed in the bibliography.

Homework problems include: Prove that the square root of 3 is irrational. Do our proofs show that $\sqrt{4}$ is irrational? Why or why not? This second question leads into a discussion of how we evaluate theorems and their proofs. Unless we can identify where in our earlier proofs the reasoning would fall apart when applied to $\sqrt{4}$, then we can not really be confident of our claim regarding $\sqrt{2}$. In addition to testing our proofs, we should be attentive to the conditions of the theorems we state and encounter. We should be able to show how removal of each one of a theorem's givens leads to a collapse of its conclusions. Consider the general theorem "if a counting number is not a perfect square then its square root is irrational." What if we do not start with a counting number? What if the counting number is a perfect square? What if it is a perfect

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40 The conclusion that an even perfect square must have an even root actually requires a lemma that the students should attempt. It does not follow from its previously proven converse (that an even squared is even).

41 To show that $2a^2 = b^2$ look at the possible rightmost digits of b^2. They are 0, 1, 4, 5, 6, or 9. Given this list, the possible last digits of $2a^2$ are 0, 2, 3, 5, 7, 8. Since the two sides are equal, only 0 and 5 are possible candidates for the last digits. In both of these cases, *a* and *b* must have a common factor of 10 or 5 which contradicts the original assumption.

cube? Theorems are understood and extended by transforming or removing hypotheses. The more we practice such analyses for theorems or generate cases to test definitions, the better our instincts become for the number and types examples which we need in order to feel confident about the meaning of a definition or the truth of a conjecture.

Two well-known corollaries to the irrationality of $\sqrt{2}$ are that the side of a square and its diagonal are incommensurable and no perfect square is twice any other. One of my students added the surprising corollary that no two consecutive series of odd numbers starting at 1 could be equal.[42] I give a mini-lecture on the historical significance of this proof in the development of Greek mathematics. This information is available in many history of math books including lectures 5 and 6 of Great Moments in Mathematics (Before 1650) by Howard Eves (published by the Mathematical Association of America). It is interesting to note that the decimal expansions of π and other irrational values have been computed to billions of places. These expansions have not settled down into a repeating pattern nor did we expect them to do so. These unfathomably long calculations, however, are examples and in no way should convince us of the ultimate behavior of π or $\sqrt{2}$. It is the proofs that should persuade us.

It is valuable for students to be exposed to classic proofs and examples of clear reasoning. They are asked to generate their own proofs periodically throughout the course. Practice with simple algebraic proofs can help students understand how to set up simple non-geometric arguments. For example, they can try proving that the square of an odd integer always leaves a remainder of one when divided by 4. Can they prove that the cube of a number less the original number is always divisible by six? These are trivial algebraic proofs, but they are a good starting point for students who do not have much exposure to symbolic proof.

The generation of a proof or other convincing argument is important to mathematical modeling. A modeler needs to be as certain as possible of the properties and behaviors of their mathematical representations in order to have confidence in the recommendations or understandings that are being drawn from the models.

Georg Cantor's Levels of Infinity

The final lesson of this unit is also the most abstract, most tangential, and most surprising. I present it as an intuition-stretching example of the beauty, unexpectedness, and elegance of mathematics. Even if it has little connection to modeling, it does add a new perspective to topics already explored. It is also important to share with our students milestones in the discipline. The

---

42 She noted that because perfect squares are sums of odd numbers ($n^2 = \sum_{i=1}^{n}(2i-1)$) and $a^2 \neq 2b^2$, then $b^2 \neq a^2 - b^2$ or $\sum_{i=1}^{b}(2i-1) \neq \sum_{i=b+1}^{a}(2i-1)$.

pairing of the ancient proof of the irrationality of the square root of two and the relatively modern work of Cantor from the late nineteenth century nicely bracket the history of the real numbers.

The class is presented with the following starting questions:

> Are there more even counting numbers or odd counting numbers? Why?
>
> Are there more integers or even integers?
>
> What is ∞? ∞+1? Is ∞+1 > ∞? Is 2·∞ > ∞?
>
> Are there more points in a unit segment or a unit square?
>
> If two sets are labeled equal in size, in what sense are they equal?

As discussion proceeds, students grapple with the paradoxes of comparing infinities. The question of how they would compare two large piles of objects leads them to Cantor's idea that two sets are equivalent if you can pair them up in a 1-to-1 fashion. Given this definition, can they prove (construct a 1-to-1 pairing) that the evens and odds are equivalent? Can they do so for the evens and the integers? I provide the definition that a set is *countable* if it can be shown to be equivalent to the counting numbers. What other sets might be countable? Are the rationals? The better part of a class period is spent trying to construct such a listing in pairs. Students propose the following pairing:

Counting	1	2	3	4	5	6	7	...
Rational	1	1/2	1/3	1/4	1/5	1/6	1/7	...

which they identify as being incomplete. Or...

Counting	1	2	3	4	5	6	7	8	9...
Rational	1	1/2	1/3	2/3	1/4	3/4	1/5	2/5	3/5...

which surprises them by counting all rationals in the unit interval but which misses all those greater than one or less than zero. They are amazed to see Cantor's arrangement, which shows that the rationals, despite being dense, are an infinite set equivalent to the counting numbers:

Counting	1	2	3	4	5	6	7	8	9...
Rational	1/1	1/2	2/1	1/3	2/2	3/1	1/4	2/3	3/2...[43]

Given this result, is it reasonable to assume that all infinite sets, dense or otherwise, are equivalent? No! It took Cantor several years before he could answer this question. He ultimately showed that the set of reals was greater than the countable sets. He went on to construct a hierarchy of uncountable sets. I present some of his further proofs to the class. The surprises are endless. The reals and rationals are both dense in each other and yet there are infinitely more of one than the other. Since the rationals combined with the irrationals make the reals, the irrationals must contribute the uncountable part. Remarkably, Cantor showed that the most

43 Each pair of vertical bars groups all fractions whose numerator and denominator sum to the same total.

familiar irrationals, such as the roots, were also countable and it was only the seemingly rare numbers such as π and e, the transcendentals, that constituted the bulk of the real numbers and made the set uncountable. The bibliography (Appendix B) includes excellent resources on this topic.

The First Quest/Problem Set

Activity: The first problem set on numbers and number properties.

Modeling Skills: Researching information; generating test cases; justification and proof; estimation; clear writing.

Audience: Secondary school onward. Younger secondary students might be given fewer problems or just the national debt essay to focus on initially.

Materials: Reference sources (preferably a library, but a variety of almanacs can substitute).

Length: A week to ten days depending, in part, on access to reference materials for the debt essay and the quantity of parallel homework assignments.

Comments: Students should be encouraged to treat the quest as a series of nightly assignments. First night, read the problems carefully and play with them enough to see if the questions are fully understood. Second night, start work on the essay and one other question. Avoid leaving the most difficult problems until the end and make sure that all problems receive at least a second reading, two separate considerations, and a check.

Because problem sets are meant to encourage depth of exploration, they cannot include every topic or skill addressed. The poster and ladder assignments provide ample evidence from an interesting task about a student's unit skills. Although the debt essay may draw on estimation skills, it does not focus on unit conversions. These skills do reappear during later activities such as the kitchen experiments described in the Functions in Context chapter.

The class has already discussed the value of working in groups. Prior to their work on the first problem set (**N5.1**), they discuss the value of individual efforts. The contrast must be made because students become accustomed to the advantages of helping each other. However, it is important to grapple with a problem on one's own. It is imperative for me to see what their individual strengths and understandings are so I can help them use those abilities to the maximum and get stronger in areas that remain a challenge. I remind them that each collected work is part of an ongoing process and that they should expect to rework the less successful parts of their problem set.

I urge students to read and re-read the problem set questions and to spend some time each day seeking solutions. Although I stress reading and writing, I also caution them that this is not a creative writing course and answers should draw on the issues discussed in class. For example, some students will answer the third question about a race to find the complete decimal expansion of π by stating that it is a waste of money. They have not realized that the task is impossible and explained why. Responses supporting the author's proposal sometime include comments such as "if pi proves not to be irrational, that would be interesting." Their failure to accept the claim that pi has been proven to be irrational should convince them of the need to prove as many claims as possible for themselves.

The tendency to trust the substance of an author's comments appears in responses to the fifth question on debt reduction proposals. The article they read includes several absurd suggestions for how to reduce the debt, none of which reflect any understanding of the enormity of the total. The only proposal that would even make a noticeable dent on the problem is the one that calls for a complete default by the government! Some answers reasonably argue that the target of some proposals is worth supporting without making the additional claim that the amount only negligibly contributes to the deficit anyway. When I return their problem sets, I distribute examples of their most creative and thoughtful answers (**N5.2**). I point out that a response that rebuts a proposal on social, political, and economic grounds is more persuasive than one that viewed the situation from a single perspective. In addition to written feedback throughout their problem sets, I provide three scores. One reflects how they did on problems 1cii and 2 which both require a proof. Another score is for their debt essay and addresses their research, writing, and estimation skills. The final score is for all of the other problems and addresses the number types and their properties. I point out to the class that I have not totaled or averaged the three, because doing so would obscure the information that each provides. I encourage them to pick a topic with a low score and to re-submit a second try at that problem or a related one provided by me. I do not want them to see problem sets as the end of a process, but as an intermediate step in which we gauge their progress and point them in the right direction for further growth.

Appendix A - Bibliography

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Gross, Fred, Morton, Patrick, and Poliner, Rachel. The Power of Numbers: A teacher's Guide to Mathematics in a Social Studies Context. Educators for Social Responsibilty, Cambridge, Massachusetts, 1993. *An excellent middle school curriculum that can be used in its totality or as a supplement to other curricula.*

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A superb, concise, and accessible introduction to Cantor's work and the first discoveries about the levels of infinities.

Dauben, Joseph. "Georg Cantor and the Origins of Transfinite Set Theory." Scientific American (June 1983): pp. 122-131. *A history of Cantor's life and work and technical presentation of his ideas.*

Butler, David and Mark Feeney. *Love Me Legal Tender: A King-sized Debt.* The Boston Globe.

Eager, Bill. "Public Speaks: Lottery Among Suggestions to End Nation's Debt Crisis." Albany Times Union (March 6, 1994).

Numeracy Links

On rounding and precision: http://www.angelfire.com/oh/cmulliss/ and http://forum.swarthmore.edu/dr.math/faq/faq.rounding.html.

On Exponential Ladders: http://physics.hallym.ac.kr/reference/scales/scales1.html. *A highly technical, but interesting set of examples for25 different units.*

Appendix B - Handouts*

- **N1.1** - Solar System Scale Model Activity Handout.
- **N1.2** - Solar System Data Handout.
- **N2.1** - Calculator Precision and Rounding Limitations Handout.
- **N2.2** - Units and Conversion Reference Guide and Practice.
- **N3.1** - *On Number Numbness*.
- **N3.2** - Ladder of Mass Fill-in Sheet.
- **N3.3** - Ladder of Mass Answers.
- **N3.4** - Ladder of Mass Follow-up Questions.
- **N3.5** - Poster Rubric
- **N3.6** - Exemplary Student Exponential Ladders.
- **N4.0** - Common Numbers Reference.
- **N4.1** - *Zero* - reading.
- **N4.2** - Questions about *Zero* and Number Category Descriptions.
- **N4.3** - Classification and Closure.
- **N4.4** - ‘Large’ and ‘Small’ Numbers.
- **N4.5** - Decimal to Fraction Conversion Techniques.
- **N5.1** - Take-home Problem Set on Numbers and Number Properties.
- **N5.2** - Sample Student Responses to the Problem Set.

* Documents are numbered within each unit and sub-unit (N3.2 is the second document of the third sub-unit of the Numbers in Context unit).

Scale Model of the Solar System

Group Members' Names:

_______________ _______________ _______________

Objective

To construct a scale model of the Solar System using the stone sphere as the Sun.

1) Prior to any calculations or discussions within your group, please record your estimates/predictions for the values of the following quantities:
   The diameter of the stone sphere: _____________ .
   The distance from the Earth to the Sun if the Sun were as big as the stone sphere ___________ .
   A common object that might represent the Earth at this scale ____________ .

2) Work with your partners to determine a method for estimating the size of the stone sphere and carry out that method. Describe your method and record your calculations in the space below.

3) Calculate the diameter and orbital distance from the Sun for each of the nine planets in this model. Add one or more moons to your model if you have the time. Show all your work and record your results on the second page of this handout.

4) Locate objects that represent the planets (at the correct scale).

5) Place your model planets at the correct distance from the sphere in the park.

6) Determine a method for judging the accuracy of your calculations and model.

The Planets	Diameter of Model Object	Distance from the Sphere	Object Chosen
Sun		0	Stone Sphere
Mercury			
Venus			
Earth			
Moon		Distance from the Earth:	
Mars			
Jupiter			
Saturn			
Uranus			
Neptune			
Pluto			

Solar System Data	Minimum Distance from the Sun (x 10^6 km)	Maximum Distance from the Sun (x 10^6 km)	Orbital Velocity (miles per second)	Sidereal Revolution (days)	Mean Semi-diameter (km)	Volume (Earth = 1) (Earth's is 1.08 x 10^{21} m^3)*	Mass (Earth = 1) (Earth's is 5.97 x 10^{24} kg)*	Density (Earth = 1) (Earth's is 5.53 gm/cm^3)	Gravity at Surface (Earth = 1) (Earth's is 9.8 m/s^2)
Sun	0	0	---	---	696100	1303730.0000	332950.0000	0.26	27.9
Mercury	46	69	29.75	88.0	2422	0.054	0.0055	0.96	0.37
Venus	107	108	21.76	224.7	6054	0.880	0.8150	0.94	0.88
Earth	147	152	18.51	365.3	6373	1.000	1.0000	1.00	1.00
Mars	206	249	14.99	687.0	3390	0.149	0.1075	0.72	0.38
Jupiter	741	815	8.12	4332.1	71240	1316.000	317.8900	0.24	2.64
Saturn	1349	1509	5.99	10825.9	60030	755.000	95.1840	0.13	1.15
Uranus	2686	2992	4.23	30676.1	24460	52.000	14.5360	0.29	1.15
Neptune	4442	4541	3.38	59911.1	25100	57.000	17.1480	0.30	1.12
Pluto	4436	7324	2.95	90824.2	1287	0.006	0.0020	0.19	0.04

* Note: these measurements are aligned at the decimal. The precision is not as great as may appear for some values.

Largest Satellites for Each Planet	Planet	Minimum Distance from planet (x 10^3 km)	Orbital Period (days)	Diameter (km)
none	Mercur	---	---	---
none	Venus	---	---	---
Moon	Earth	384.5	27.32	
Phobos	Mars	9.4	0.32	21
Ganymede	Jupiter	1070	7.16	5260
Titan	Saturn	1221	15.95	5550
Titania	Uranus	436.3	8.70	1610
Triton	Neptun	354	5.88	2700
Charon	Pluto	19.1	6.39	1200

Conversion Equivalents
1 kilometer = 1000 meters = .6214 miles.
5280 feet = 1 mile.
1 meter = 3.28 feet.
1 kilogram = 1000 grams = 2.19 pounds.

Calculator Correctness   Name = ___________________

Do this operation with your calculator (do not hit the = or enter key until the end):	Copy the calculator's result in standard or scientific notation:	Is the calculator's answer exactly correct? (Yes/No)
A) $2 \div 3$		
B) $(5 \times 10^7) \times (3 \times 10^4)$		
C) 12345678×12345678		
D) $15 + 5 \times 9$		
E) $41 \div 8$		
F) $10^{50} - 10^{20}$		
G) $1 \div 3 \times 3 - 1$		
H) $1 \div 999999999999$		
I) 9999999 - 0.0000001		

For each of the calculations above which you said was not correct, explain what was wrong and why you think it was wrong and why you think the calculator did what it did. (Use the back of the handout for additional writing space).

Units and Conversion Reference Guide

The three most familiar physical quantities, in terms of which most other quantities are defined, are length (L), time (T), and mass (M). The study of units is called dimensional analysis and a variable is defined in terms of its units or dimensions. For example, area is the product of two lengths (L^2) and units of area will typically (although not always) be the square of a linear unit. Likewise, volume is a three-dimensional quantity and is the cube of a length (L^3), although units of volume such as quart and liter may not make the connection apparent.

Physical quantities combining time, length, and mass, their dimensions, and example units follow:

Speed = change in distance during a given amount of time = $\frac{L}{T}$. Car speeds are reported in miles or kilometers per hour.

Acceleration = change in speed over time = $\frac{\frac{L}{T}}{T} = \frac{L}{T^2}$. Earth's gravity accelerates objects at sea level 9.8 meters per second per second or per second squared.

Density = mass per unit of volume = $\frac{M}{L^3}$. The density of water at 4°C is 1 g/cm^3.

Force = mass times acceleration = $\frac{ML}{T^2}$. The metric unit of force, the newton, is kilogram·meters per second squared.

Energy = force exerted over a distance = $\frac{ML}{T^2} \cdot L = \frac{ML^2}{T^2}$. The joule's units are newton·meters.

Pressure = force exerted per unit of area = $\frac{\frac{ML}{T^2}}{L^2} = \frac{M}{LT^2}$.

Common Unit Abbreviations

Lengths	Mass	Volume
in - inch or inches	g - grams	l - liter
ft - foot or feet	lb - pound	oz - ounces
m - meters	oz - ounces	Time

Quantities

Million -	1,000,000
Billion -	1,000,000,000
Trillion -	1,000,000,000,000
Quadrillion -	1,000,000,000,000,000

s - seconds Quintillion - 1,000,000,000,000,000,000

Common Metric Prefixes

pico (p) = 10^{-12} or one trillionth of.	tera (T) = 10^{12} or one trillion of.
nano (n) = 10^{-9} or one billionth of.	giga (G) = 10^{9} or one billion of.
micro (μ) = 10^{-6} or one millionth of.	mega (M) = 10^{6} or one million of.
milli (m) = 10^{-3} or one thousandth of.	kilo (k) = 10^{3} or one thousand of.
centi (c) = 10^{-2} or one hundredth of.	hecto (h) = 10^{2} or one hundred of.
deci (d) = 10^{-1} or one tenth of.	deka (da) = 10^{1} or ten of.

Each of the above prefixes is used in front of any of the standard metric units. For example, a kilometer, abbreviated km, is one thousand meters. A picogram, pg, is one trillionth of a gram. Conversion factors can be written using each prefix. Because a millimeter is equal to one thousandth of a meter, you can write: $\frac{1\ \text{mm}}{\frac{1}{1000}\ \text{m}} = \frac{1000\ \text{mm}}{\text{m}}$. If you wanted to convert $3.58 \cdot 10^{5}$ millimeters to megameters, you could write $3.58 \cdot 10^{5}\ \text{mm} \cdot \frac{1\ \text{m}}{1000\ \text{mm}} \cdot \frac{1\ \text{Mm}}{1{,}000{,}000\ \text{m}} = 3.58 \cdot 10^{-4}$ Mm. Each conversion factor canceled out a unit you did not want in favor of one that you did want or one that linked you to the one that you wanted. Alternatively, a look at the chart above reveals that the prefixes milli- and mega- are nine powers of ten apart and the factor $\frac{1\ \text{Mm}}{10^{9}\ \text{mm}}$ will complete your conversion in one step.

Conversion Values Within and Between the Metric and British systems:

Length

$\frac{12\ \text{in}}{\text{ft}}$ $\frac{3\ \text{ft}}{\text{yard}}$ $\frac{5280\ \text{ft}}{\text{mile}}$ $\frac{2.54\ \text{cm}}{\text{in}}$ $\frac{3.281\ \text{ft}}{\text{m}}$ $\frac{9.4605 \cdot 10^{12}\ \text{km}}{\text{light} - \text{year}}$

Area

$\frac{10000\ \text{m}^2}{\text{hectare}}$ $\frac{43560\ \text{ft}^2}{\text{acre}}$

Time

$\frac{365.242198781\ \text{days}}{\text{year}}$

Mass and Weight*

$\frac{2.2046\ \text{lb}}{\text{kg}}$ $\frac{16\ \text{oz}}{\text{lb}}$ $\frac{2000\ \text{lb}}{\text{ton}}$ $\frac{1000\ \text{kg}}{\text{metric ton}}$

Volume

$\frac{1\ \text{litre}}{1000\ \text{cm}^3}$ $\frac{1.056688\ \text{litres}}{\text{quart}}$ $\frac{32\ \text{oz}}{\text{quart}}$ $\frac{4\ \text{quarts}}{\text{gallon}}$ $\frac{8\ \text{oz}}{\text{cup}}$ $\frac{2\ \text{cups}}{\text{pint}}$

* Although a conversion factor is provided for pounds and kilograms, they are not the same kind of unit. Kilograms are a unit of mass while pounds are a weight. Weight is a measure of a gravitational pull on an object's mass and is, therefore, a force. The conversion factor between them is based on the weight that a given amount of mass has at the surface of the Earth. On planets with a different gravitational pull, an object would have the same mass, but a different weight.

Unit Conversion Exercises

Examples:

Convert 10 gallons into cups.

One solution, using only the conversion factors listed in the handout, is:

$$10 \text{ gallons} \cdot {}^{4\text{ quarts}}/_{\text{gallon}} = 40 \text{ quarts} \cdot {}^{32\text{ oz}}/_{\text{quart}} = 1280 \text{ oz} \cdot {}^{1\text{ cup}}/_{8\text{ oz}} = 160 \text{ cups.}$$

Note that the reciprocal of 8 oz/cup was used in order to cancel out ounces and introduce cups.

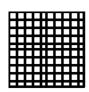
The dyne is a metric unit of force equivalent to ${}^{g\cdot cm}/_{s^2}$. How many newtons is one dyne?

The units for the newton are ${}^{kg\cdot m}/_{s^2}$, so the grams need to be replaced with kilograms and the centimeters need to be replaced with meters:

$1 \text{ dyne} = 1\, {}^{g\cdot cm}/_{s^2} \cdot 1\, {}^{kg}/_{1000\, g} = 0.001\, {}^{kg\cdot cm}/_{s^2} \cdot 1\, {}^{m}/_{100\, cm} = 0.00001\, {}^{kg\cdot m}/_{s^2} = 10^{-5}$ newtons.

This conversion tells us as well that there are 100,000 dynes per newton.

Units of length serve as a basis for units of area and volume. You cannot, however, simply swap the two. For example, 1 centimeter equals 10 millimeters, but, as the diagram below illustrates, 1 cm^2 equals 100 mm^2.



Attention to units confirms this picture. Starting with 1 cm^2, we need to multiply by ${}^{10\text{ mm}}/_{\text{cm}}$ twice (or $({}^{10\text{ mm}}/_{\text{cm}})^2$ once) in order to cancel out both cm factors.

Practice Problems

A) A light-year is the distance light travels in one year. How many miles is one light year?

Starting with light's velocity of 1 light-year per year, find its speed in meters per second.

B) A handy rule of thumb is that 5 miles is approximately equal to 8 kilometers (note that on a speedometer the 50 mph notch is close to the 80 km/h one). Verify this approximation using the conversion factors from your reference handout.

C) My cereal box advertises that it contains only 10 grams of sugar per 1 ounce serving. How many grams are there in an ounce and what percent of my nutritious breakfast is sugar?

D) How many centimeters are in a kilometer? How many seconds are in a day? How many tons are in an ounce?

E) What is your mass in kilograms? What is your height in meters?

F) How many square inches are in a square foot? In a square mile? In a square centimeter? In a cubic centimeter? How many acres are in a square mile?

G) Find a conversion factor between cubic yards and cubic feet. How many cubic feet are in a cubic mile?

H) An acre·inch is the amount of water needed to cover an acre of land to a depth of one inch. How many cubic feet of water is one acre·inch ?

I) Convert 60 miles per hour into feet per second. What percent of the speed of light is 60 mph?

J) The joule's units are $kg \cdot m^2/sec^2$. The erg's units are $grams \cdot cm^2/sec^2$. How many joules are in one erg?

K) The density of air at sea level in 1.3 kg/m^3. How many pounds would a cubic mile of air weigh (our atmosphere does become thinner with increasing altitude, but assume an unchanging density).

L) How many dm^3 equal one liter? What is your approximate volume in liters?

M) Goodyear claims that the Aquatred tire's "Aquachannel sweeps away over a gallon of water every second, or 396 gallons a mile." Can you derive any new information and draw any conclusions about this tire's performance?

Solutions to the Unit Conversion Practice Problems

A) $5.88 \cdot 10^{12}$. 299,800,000 meters per second (or roughly seven times around the Earth per second!).

B) $5 \text{ miles} \cdot \frac{5280 \text{ ft}}{\text{mile}} = 26400 \text{ ft} \cdot \frac{1 \text{ m}}{3.281 \text{ ft}} = 8046 \text{ m} \cdot \frac{1 \text{ km}}{1000 \text{ m}} = 8.046 \text{ km}$

C) $1 \text{ ounce} \cdot \frac{1 \text{ lb}}{16 \text{ oz}} \cdot \frac{1 \text{ kg}}{2.2046 \text{ lb}} \cdot \frac{1000 \text{ g}}{1 \text{ kg}} = 28.35 \text{ grams}$. The cereal is 35% sugar.

D) 100,000 cm/km. 86,400 sec/day. 1/32,000 tons per ounce.

E) 81 kg and 1.88 m, for me anyway.

F) 144 square inches per square foot. 4,014,489,600 in^2 per $mile^2$. 0.061 in^2 per cm^2. Infinite because the third dimension is 0 or just disqualify the question. 640 acres/sq. mile.

G) 27 cubic feet per cubic yard. $(5280)^3$ $1.47 \cdot 10^{11}$ ft^3 in a $mile^3$.

H) 3630 cubic feet.

I) 88 ft/sec. A car traveling 60 mph is traveling at roughly one ten millionth the speed of light.

J) 0.0000001. There are 10 million ergs in one joule.

K) $1.2 \cdot 10^{10}$ lb per $mile^3$ of air. (Air is light, but cubic miles are big!). For comparison, note that air pressure at sea level is 14.7 lb/in^2 which is equivalent to 1 atmosphere and gives the weight of the column of air above a square inch of land.

L) A liter is equal to one deciliter. Can you use my weight (see problem E) to estimate my volume? What is the conversion factor for human "stuff" between mass and volume?

M) $\frac{1 \text{ gal}}{1 \text{ sec}} \cdot \frac{1 \text{ mile}}{396 \text{ gal}} = \frac{1 \text{ mile}}{396 \text{ sec}} \cdot \frac{3600 \text{ sec}}{\text{hour}} = 9.1 \frac{\text{mile}}{\text{hr}}$. Personally, I am not too worried about how my tires handle water at this speed.

Exponential Ladder of Mass   Name ________________________

Place one item from the list at the bottom of the page onto the line of the chart closest to the weight of that item. Exactly one item belongs on each line. Try to put them in proper order. Pay attention to the units. Measurements appearing on the same line are equal.

$^{1}/_{3,000,000,000}$ of an ounce	10^{-8} g	0.01 µg	________________
	10^{-7} g	0.1 µg	________________
	10^{-6} g	1µg (microgram)	________________
$^{1}/_{3,000,000}$ of an ounce	10^{-5} g	10 µg	________________
	0.0001 g	0.1 mg	________________
$^{1}/_{500,000}$ of a pound	0.001 g	1 mg (milligram)	________________
$^{1}/_{3,000}$ of an ounce	0.01 g	10 mg	________________
	0.1 g	100 mg	________________
	1 gram	1000 mg	________________
1/3 of an ounce	10 g	0.01 kg	________________
3.5 ounces (oz)	100 g	0.1 kg	________________
2.2 pounds (lb)	1,000 g	1 kg	________________
22 lb	10,000 g	10 kg	________________
220 lb	100,000 g	100 kg	________________
1.1 tons	1,000,000 g	1,000 kg (1 metric ton)	________________
11 tons	10^{7} g	10,000 kg	________________
110 tons	10^{8} g	100,000 kg	________________
1100 tons	10^{9} g	1,000,000 kg (or 1 gigagram)	________________

A grain of wheat

The tyrannosaurus rex dinosaur

An average-sized cell in your body

A filled book bag

A pair of chicken eggs

A moth egg

A dust particle

A blue whale

A 5 karat diamond

A small mouse

An amoeba or paramecium

The smallest fish known

An automobile

A droplet of mist

The heaviest tree (a sequoia in California)

A human egg

A quart of milk

A large adult

The Answers for the Exponential Ladder of Mass

An average-sized cell in your body (0.00001 mg)

A dust particle

A human egg (0.001 mg)

An amoeba or paramecium

A droplet of mist

The smallest fish known or 1/50 of a drop of water (1 mg)

A moth egg

A grain of wheat

A 5-karat diamond (1 g)

A small mouse

A pair of chicken eggs

A quart of milk (1 kg)

A filled book bag

A large adult

An automobile (1000 kg)

The tyrannosaurus rex dinosaur

A blue whale

The heaviest tree (a sequoia in California) (1,000,000 kg)

Information for this exponential ladder comes primarily from: Asimov, Isaac. The Measure of the Universe. Harper and Row, New York, 1983.

Questions on the Exponential Ladder of Mass Name ____________________

Use the values on the answer sheet for your calculations.

How many times bigger is a moth egg than a human egg?

How many times bigger is a sequoia than a large adult?

What is the difference in mass between a moth egg and a human egg?

What is the difference in mass between a sequoia and a large adult?

How many chicken eggs weigh as much as a book bag?

Approximately how many cells are in a human's body?

Why is a geometric (exponential) ladder better than an arithmetic (linear) one?

Why are whales able to grow larger than any land animal?

Why are trees able to grow larger than any land or ocean animal?

NAME ______________________________ POSTER TOPIC ______________________

The poster provides a
comparison of **extreme** quantities.
The information is
accurate (bibliographic information on back)
important and/or **interesting**
The poster
accurately and **graphically illustrates**
the relationship between the quantities
The graphics are
carefully executed, original,
attractive, colorful, artistic,
eye-catching, engaging
The Poster is
sturdy in its construction.

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Exponential Ladder of Length

10^{-18}	1 x 10-18	Diameter of an Electron or Quark[8]
10^{-17}		
10^{-16}		
10^{-15}	1×10^{-15}	Diameter of Proton or Neutron (One Fermi)[8]
10^{-14}	1×10^{-14}	Diameter of Atomic Nucleus[8]
10^{-13}		
10^{-12}	2.426×10^{-12}	Compton Wavelength[4]
10^{-11}	$.529 \times 10^{-11}$	Bohr Atomic Radius[2]
10^{-10}	1×10^{-10}	A Molecule of Water…or One Angstron[12]
10^{-9}	3.4×10^{-9}	One Complete Twist of DNA Double Helix[14]
10^{-8}	1×10^{-8}	Wavelength of Ultraviolet Light[2]
10^{-7}	1×10^{-7}	Thickness of Cell Membrane[12]
10^{-6}	1×10^{-6}	Length of Typical Bacteria[22]
10^{-5}	1×10^{-5}	Length of Typical Cells[12]
10^{-4}	1×10^{-4}	Width of a Human Hair[23]
10^{-3}	.001	Length of Typical Piece of Sand[5]
10^{-2}	.01	Radius of Nickel[15]
10^{-1}	.9	Side of a 3 ½ Floppy[15]
10^{0}	1	A Meter Stick[9]
10^{1}	12	Half Court of Basketball Court[9]
10^{2}	91 (100 Yards)	Length of American Football Field[9]
10^{3}	979	Height of Angel Waterfalls, Venezuela (Highest in World)[11]
10^{4}	10911	Depth of Challenger Deep in Pacific Ocean[16]
10^{5}	98652	Length of Kiel Canal, Germany (10 Mi. Longer than Panama Canal)[11]
10^{6}	1,013,886	Length of Gila River (Flows into Colorado River)[16]
10^{7}	12,729,911	Diameter of the Earth[4]
10^{8}	120,000,000	Diameter of Saturn[21]
10^{9}	1,390,473,216	Diameter of the Sun[4]
10^{10}	5.8×10^{10}	Distance from Sun to Mercury[7]
10^{11}	1.496×10^{11}	Distance from Sun to Earth[6]
10^{12}	5.79×10^{12}	Distance from Sun to Pluto[7]
10^{13}	1×10^{13}	Diameter of Solar System[18]
10^{14}		
10^{15}		
10^{16}	9.46×10^{15}	One Light Year…(4.3 Light Years to Alpha Centauri)[9]
10^{17}	1×10^{17} (11 Light Years)	Distance from 61 Cygni to Earth[1]
10^{18}	9.46×10^{17} (100 Lt-Yr)	Distance to Alpha Carnae (Canopus) from Earth[1]
10^{19}	7.56×10^{18} (800 Lt-Yr)	Distance to Rigel in Orion from Earth[1]
10^{20}	2.83×10^{20} (30,000 Lt-Yr)	Distance from the Sun to the Center of the Milky Way[1]
10^{21}	9.46×10^{20} (100,000 Lt-Yr)	Length of the Milky Way[13]
10^{22}	1.89×10^{22} (2 Million Lt-Yr)	Distance to M 31 Galaxy in Andromeda from Earth[1]
10^{23}	3.78×10^{23} (40 Million Lt-Yr)	Distance to Spiral Galaxy M 104 from Earth[1]
10^{24}	9.9×10^{23} (100 Million Lt-Yr)	Distance to Black Hole in NGC 4261 from Earth[16]
10^{25}	1.7×10^{25} (1.7 Billion Lt –Yr)	Distance to Boötes Galaxy from Earth[1]
10^{26}	1.4×10^{26} (15 Billion Lt-Yr)	Distance to Quasars (Close to Edge of Ever Expanding Universe)[18]

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N4.0 Common Numbers

N	N^2	N^3	N	3 N	1/N	Primes		Exact	Other Form
1	1	1	1	1	1	2		3/2	0.8660
2	4	8	1.41421	1.25992	0.5	3		2/2	0.7071
3	9	27	1.73205	1.44225	0.33333	5		1/ 2	0.7071
4	16	64	2	1.5874	0.25	7		1/ 3	0.5773
5	25	125	2.23607	1.70998	0.2	11		e	2.71828
6	36	216	2.44949	1.81712	0.16667	13		ϕ	1.61803
7	49	343	2.64575	1.91293	0.14286	17		1/ϕ	0.61803
8	64	512	2.82843	2	0.125	19		i	(-1)
9	81	729	3	2.08008	0.11111	23		pi	3.14159
10	100	1000	3.16228	2.15443	0.1	29		pi/2	1.5708
11	121	1331	3.31662	2.22398	0.09091	31		pi/3	1.0472
12	144	1728	3.4641	2.28943	0.08333	37		pi/4	0.7854
13	169	2197	3.60555	2.35133	0.07692	41		pi/6	0.5236
14	196	2744	3.74166	2.41014	0.07143	43		log 2	0.30103
15	225	3375	3.87298	2.46621	0.06667	47		log 3	0.47712
16	256	4096	4	2.51984	0.0625	53		log 5	0.69897
17	289	4913	4.12311	2.57128	0.05882	59		ln 2	0.69315
18	324	5832	4.24264	2.62074	0.05556	61			
19	361	6859	4.3589	2.6684	0.05263	67			
20	400	8000	4.47214	2.71442	0.05	71			
21	441	9261	4.58258	2.75892	0.04762	73			
22	484	10648	4.69042	2.80204	0.04545	79			
23	529	12167	4.79583	2.84387	0.04348	83			
24	576	13824	4.89898	2.8845	0.04167	89			
25	625	15625	5	2.92402	0.04	97			
26	676	17576	5.09902	2.9625	0.03846	101			
27	729	19683	5.19615	3	0.03704	103			
28	784	21952	5.2915	3.03659	0.03571	107			
29	841	24389	5.38516	3.07232	0.03448	109			
30	900	27000	5.47723	3.10723	0.03333	113			
31	961	29791	5.56776	3.14138	0.03226	127			
32	1024	32768	5.65685	3.1748	0.03125	131			
33	1089	35937	5.74456	3.20753	0.0303	137			
34	1156	39304	5.83095	3.23961	0.02941	139			
35	1225	42875	5.91608	3.27107	0.02857	149			
36	1296	46656	6	3.30193	0.02778	151			
37	1369	50653	6.08276	3.33222	0.02703	157			
38	1444	54872	6.16441	3.36198	0.02632	163			
39	1521	59319	6.245	3.39121	0.02564	167			
40	1600	64000	6.32456	3.41995	0.025	173			

Questions on the Essay "Zero" by Constance Reid

When and why was a symbol for zero first invented? Who invented it?

In your own words, explain why 0÷4 is a real number.

In your own words, explain why 4÷0 is not a real number.

Record any questions that you may have regarding the reading.

Beyond Zero...

Please provide examples with your answers.

Find at least two properties (differences, behaviors) of the integers that the reals do not have.

Find at least two properties of the reals that the integers do not have.

Number Sets (**their symbols**) and Descriptions

Natural, Counting (**N**): 1, 2, 3, 4, 5, 6, …

Whole: 0 added to the natural numbers (0, 1, 2, 3, 4, …).

Integer (**Z**): The whole numbers along with their opposites (…, -4, -3, -2, -1, 0, 1, 2, 3, 4, …).

Rational (**Q**): Numbers which can be written as the ratio of two integers. For example, 7 and 2.1 are rational because they can be written $^{7}/_{1}$ and $^{21}/_{10}$. Has a terminating or repeating decimal expansion.

Irrational: A real number that is not rational (cannot be expressed as the ratio of two integers). Irrationals have infinite, non-repeating decimal representations (e.g., ð = 3.14159265358…).

Algebraic: Any number that can be the solution to an equation P(x) = 0 where P(x) is a polynomial with rational coefficients. ($^{2}/_{9}$ because of $9x - 2 = 0$, $\sqrt{3}$ because of $x^2 - 3 = 0$).

Transcendental: Any real number that is not algebraic (e.g., ð, 0.12345678910111213141516171…)

Real(ê): Any number expressible in decimal form.

Imaginary: A square (or even) root of a negative real number ($\sqrt{-1} = \mathbf{i}$, $\sqrt{-9} = \sqrt{9} \cdot \sqrt{-1} = 3\mathbf{i}$).

Complex (**C**): the sum of a real and imaginary number (a + b**i**, with a and b both real). Either component can be zero thus permitting the reals and imaginaries to be classified as subsets of the complex set.

Positive: A real number greater than zero.

Negative: A real number less than zero.

Prime: A counting number with only 1 and itself as factors. 1 is not defined as a prime so that all counting numbers will have a unique prime factorization.

Composite: A counting number (other than 1) that is not prime.

Real Numbers

For each number below, identify as many of the following categories that apply: Counting (N), whole, integers (Z), rationals (Q), irrationals, reals (R).

$^{13}/_{8}$

$\sqrt{35}$

$\sqrt{36}$

$\sqrt{-9}$

-321

$\sqrt{15} \cdot \sqrt{15}$

ð

3.14159

.010267267267267267…

3.122333444455555666666...

$-^{12}/_{3}$

Name a situation (either abstract or real world) where you would want to limit consideration to

whole numbers

integers

rationals

A set is closed under (or with respect to) an operation if applying the operation to members of the set always results in a member of the set. For each of the following problems, give a counter-example if you claim a set is not closed or provide a more general reason if you claim that it is closed.

Are the counting numbers closed under subtraction? Multiplication?

Are the integers closed under multiplication? Division?

Are the rationals closed under multiplication? Division? Can you prove either answer?

Are the odd integers closed under subtraction? Squaring? Raising to powers (i.e., is every odd integer to an odd integral power another odd integer)?

Are the irrationals closed under addition? Multiplication?

Find an operation under which the even integers are closed and one under which they are open.

Define a set of numbers which is closed under squaring but open under multiplication.

Numbers Large and Small

Numbers come in an infinite range of sizes and yet we often speak of 'small' and 'large' numbers. One definition (when working with positive numbers only) for these terms could be that a number is *small* if the multiplication of it with another number yields a result that is smaller than that other number. A number is *large* if the multiplication of it with another number yields a result that is greater than that other number.

Are there any positive numbers that are neither small nor large? If so, what are they? If not, justify your claim.

What is true about all small numbers when put into scientific notation?

What is common to large numbers expressed in scientific notation?

Fill in each box in the multiplication table below with SMALL, LARGE, or VARIES. Only use SMALL or LARGE if the product indicated must always be of that type. If the product can be either depending on the numbers chosen, write VARIES and give examples illustrating each of the possibilities.

Multiplication $\boxtimes$	Small	Large	Numbers greater than 100
Small			
Large			

What variations can you propose to extend this exploration?

Converting Between Fraction and Decimal Forms

To convert a terminating decimal into a fraction, place the number with the decimal point removed over ten raised to the power equal to the number of decimal places in the original value.

$$31.537 = \frac{31537}{10^3} = \frac{31537}{1000}$$

All place values have been converted into the lowest one present, thousandths. This step is a unit conversion: $\frac{31.537}{1} \cdot \frac{1000}{1000} = \frac{31537}{1000}$. The decimal portion may be converted alone resulting in the mixed fraction 31 $^{537}/_{1000}$. Note that the label "improper fraction" for the first result reflects an unwarranted bias. It would be easier to square the improper form than the "proper" one.

To convert a fraction into decimal form, you can grab a calculator and divide. However, the precision limits of the calculator will yield incomplete conversions for fractions such as 1/17 or 1000000/1000001 (see problems 1 and 2). Even the calculator result for $^1/_3$, 0.3333333333, which is suggestive of a forever repeating pattern, falls far short of the target. How can we tell when, or if, a decimal representation will repeat? Although the long division algorithm* is used much less often since the proliferation of hand-held calculators, it still provides crucial insight into this question which the calculator, whose methods of arriving at results are hidden from us, obscures. Dividing 1 by 3 produces after the first subtraction a remainder of 1. Immediately we see that the same subtractions and remainders will forever produce additional 3's in the decimal expansion. In the second example below, dividing 1 by 13 produces the remainder 1 after 5 steps (and 6 decimal places). Since 1 was the starting dividend, the cycle of steps will repeat themselves every 6 places and $^1/_{13} = .\overline{076923}$.

```
        0.33…
      3)1.00…
         9
         10
          9
          1
```

```
        0.07692307…
     13)1.00000000…
          91
           90
           78
           120
           117
             30
             26
              40
              39
              100
               91
                90
```

What assures us that this repetition will always appear when we long divide? When the divisor is 13, a multiple of 13 is chosen at each step such that the difference between that multiple and the remaining dividend is less than 13. The possible remainders, therefore, are 0, 1,

---

* An algorithm is a sequence of steps for accomplishing a task. Recipes are algorithms for producing food, directions are algorithms for getting somewhere, and the long division algorithm tells us how to divide two terminating real numbers.

2, 3, 4, 5, ..., 11, and 12. If a zero appears, then the decimal is complete. If not, then by the thirteenth step, one of the twelve positive remainders will have to appear a second time* and the repetitions will begin. This same line of reasoning applies to any division involving two integers. There will be a finite number of remainders and termination or repetition will be inevitable.

It was demonstrated above that all rational numbers will be represented as a repeating or terminating decimal. Is the converse situation also true? That is, do all terminating and repeating decimals represent rational numbers? Conversion of terminating decimals to fractions is discussed above. The problem with repeating decimals is that it is difficult to write them out to even calculate with them. That is the sticking point, so that is what needs to be overcome. The solution is to get all of those repeating digits to cancel out. The technique for achieving this cancellation is to subtract the number from a decimal shifted version of itself:

Call the original number N:	$N = 3.27272727...$
Multiply by a power of ten that lines up the digits again:	$100N = 327.27272727...$
Subtract the original equation:	$\underline{\quad N = \quad 3.27272727...}$
No more infinite decimals:	$99N = 324$
Solve for N:	$N = {}^{324}/_{99}$ or $3\,{}^{3}/_{11}$
Check:	Do the division to see that ${}^{3}/_{11} = .272727....$

The appropriate power of ten will shift the decimal point the same number of places as are repeating.

So repeating and terminating decimals and rational numbers are simply two different ways to represent the same quantities. How do we know that a non-repeating number such as 0.1011011101111011111… is irrational? Because if it could be written as the ratio of two integers then long division would yield a repeating decimal result. It is not as easy, however, to start with a number such as ð, and prove that its decimal expansion will not repeat.

---

* This claim is based on the pigeonhole principle: If there are more pigeons than pigeonholes and each pigeon is in a pigeonhole, then at least one hole has to have more than one pigeon in it. In the division example, what represents the pigeons and what represents the pigeonholes?

Problems

1) Convert each of the following to decimals.
(Use a calculator only to check your results.)

a) 3/8

b) 4/7*

c) $^{1}/_{17}$

d) $^{13}/_{6}$

e) $\sqrt{7}$

2) Convert each of the following to fractions.
(Use a calculator only to check your results.)

a) 3.07

b) $.\overline{12}$

c) $.\overline{2}$

d) $.\overline{9}$

3) When I divide 1000000 by 1000001, my calculator displays .999999. But .999999 equals 999999/1000000 which is smaller than the original fraction. What is the correct decimal expansion for 1000000/1000001? How much smaller is 999999/1000000 than 1000000/1000001?

4) Long division allows us to divide terminating numbers such as $^{2.3}/_{1.15} = 2$. You cannot write out the long division problem for $.\overline{309} / .\overline{74}$, so a different method is needed. Find and carry out this method.

5) What patterns are there to the number of repeating decimals needed for different denominators? Can you predict how many will be in the decimal expansion of $^{1}/_{N}$?

---

* Looking at the expansions for 1/7,2/7, and 3/7 first, can you predict the expansions for 4/7, 5/7, and 6/7?

Mathematical Modeling Problem Set   Name ___________________________

You must do these problems by yourself. You may consult your notes, your textbook, your graphing calculator, and Mr. Abrams only for the first four problems. For the fifth problem, you may consult additional written resources. For each of the problems, use an appropriate blend of English and mathematics to explain your ideas. Read carefully and be sure that your answers incorporate the issues explored by the class.

1) Answer yes or no for each of the following. Justify your answer.
   a) Are the odd integers closed under the operation of squaring?

   b) Are the numbers r, with $0 < r < 100$, closed under taking the reciprocal?

   c) i) Is the set $S = \{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$ dense? Explain.
      ii) Is S closed under multiplication? Give a counter-example or prove your answer.

2) Prove that $\sqrt{6}$ is irrational or (more challenging) that $\log_{10}2$ is irrational (it is also transcendental).
   Explain all steps and reasoning clearly.

3) The July 1, 1990 New York Times included a letter to the editor titled "Don't Let Japan Open a Pi Gap with the U.S." in which the author wrote of his distress that Japanese computer scientists had recently calculated the decimal expansion of ð to more places (more than a hundred million!) than had ever been previously computed. He went on to urge that the United States embark on a massive project to determine "the entire decimal expansion of ð" before anyone else. Discuss the author's proposal and offer as mathematically relevant a response as you can to his concern.

4) Give as large (most elements) a finite set as you can create that is closed under multiplication.

5) Read the article *Love me legal tender* for background information and then read ***Public Speaks*** which reports citizens' ideas for eliminating or easing the national debt.
Write an essay critiquing at least <u>two</u> of the proposals which seem more reasonable to you.
For each proposal:
   a) Identify the idea (determine what claims are made or implied).
   b) Research any information you need to evaluate the usefulness of the proposal (provide bibliographic information).
   c) Detail your calculations and comparisons.
   d) Discuss what impact implementing the plan would have on eliminating or substantially reducing the national debt. Is the idea worthy of consideration?

Bonus Questions

A) At 1 minute to midnight, the balls numbered 1 through 10 are put into an urn and ball number 1 is taken out. At half a minute to midnight, the balls numbered 11 through 20 are put into an urn and ball number 2 is taken out. At one quarter of a minute to midnight, the balls numbered 21 through 30 are put into an urn and ball number 3 is taken out. This pattern is continued. How many balls are left in the urn at midnight? Which balls are left in the urn at midnight? Answer both questions and discuss the problem. Do not feel obliged to settle on a particular answer. Don't focus on the physical impossibility of moving balls so quickly.

B) Write a careful numerical analysis of the physical and economic tasks facing the tooth fairy.

Selected Answers to Numbers Take-home Quest

1a) Yes. For the reason, you can decide whether the claim that an odd times an odd requires its own proof.

1b) No. All reals between 0 and $^{1}/_{100}$ serve as a counterexample.

1ci) No. There are no additional elements of S between any adjacent pair of elements (e.g., no element of S falls between $^{1}/_{2}$ and $^{1}/_{3}$).

1cii) Yes. Two fractions with 1 as a numerator (unit fractions) are being multiplied so the numerator will remain 1. Two counting numbers are being multiplied in the denominators and the counting numbers are closed under multiplication so the denominator will remain a counting number. Therefore, the result will be a unit fraction and an element of S.

2) Proofs that $\sqrt{6}$ or $\log_{10}2$ are irrational require that claims of a contradiction be clearly supported. To prove that $\log_{10}2$ is irrational, assume that it is rational. Set $\log_{10}2 = p/q$, convert to exponential form ($10^{p/q} = 2$), raise both sides to the q$^{\underline{th}}$ power, and show that the factors of 5 and 2 do not work out. Generalize to other $\log_b n$.

4) Typical answers include: all of the numbers between 0 and 1, which is not a finite set, but which is closed under multiplication; the set {0,1}, which works but is small; sets such as {0, 0.5, 1, 2}, which would work if elements could only be multiplied by different elements (but they can be multiplied by themselves under the definition of closure so 2 x 2 = 4 is a counterexample); And { -1,0,1} which is as good as it gets using just real numbers. The student exemplar includes a creative effort involving i and infinity. Subsets of this effort, {0, 1, -1, i, -i} and {0, 1, -1, , - }, would also work.

Only students who are quite familiar with complex numbers and their properties and who steer their thinking away from the reals as a barrier have a chance to discover that there is no largest finite set closed under multiplication. The vertices of a regular n-gon centered at the origin of the complex plane with one vertex at 1 + 0i will produce a closed set with n elements (e.g., an octagon yields the set $\{1, \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i, i, -\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i, -1, -\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i, -i, \frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\}$. Since any n can be chosen, the set can be as large as desired.

Bonus Questions

A) For any given ball n, its turn will arrive and it will be removed at $\frac{1}{2^{n-1}}$ minutes before midnight. So there will be no balls left because any ball you can name will be gone. On the other hand, at the same moment that the nth ball departs, there are $9n$ balls left in the urn with more and more of them as midnight approaches. This problem is known as a supertask. For more information on this topic, see the wonderful book aha! Gotcha: Paradoxes to puzzle and delight by Martin Gardner (W. H. Freeman and Company, New York , 1981).

B) Students might want to determine the typical number of teeth lost in a day or year, the miles (and miles per hour!) the fairy would need to travel, the weight of the money and teeth weighing them down on this journey, etc.

PUBLIC SPEAKS Albany, N.Y., Sunday, March 6,1994 **Times Union**

Lottery among suggestions to end nation's debt crisis

BY BILL EAGER
Staff writer

Bob Nevew bets we could gamble the national debt away, $2 at a time.

He calls his idea Patriot Payoff, and based on revenues generated by state lotteries nationwide, a national lottery could generate hundreds of millions of dollars a week, Nevew figures.

“Half of the earnings, half of the giveaway, every week would go to pay down the debt and half would go to winnings,” said Nevew, who proposed the idea for a marketing class at the University at Albany, where he is completing a master's degree in business administration. “You could pay off the debt fairly quickly, within a 5- to 6-year period.”

Nevew and about 40 other people called the *Times Union* last week with their ideas for reducing the deficit last week following the five-part series, “America's Debt Drain,” which ended on Thursday.

Nevew's plan was certainly among the most creative but hardly the most outrageous. While some callers discussed aid to Ukraine, rising entitlements or immigration in some depth, others boiled their feelings into a few simple ideas.

“Get everybody to go back to Philadelphia, sit down and start the whole thing all over again,” said one frustrated man. “Secondly, change the currency. Thirdly, we can all sit back and wait till it all collapses.”

Added a woman, “I'd have a national garage sale and sell all the treasures in the Smithsonian.”

Reflecting polls conducted on the nation's deepest ills, for most callers the $4.5 trillion debt is cause for anger and immediate action, not all of it pleasant.

One man said Congress should cut its own salary, as punishment for overspending, until the deficit is resolved. Others want immigration - and especially foreign aid - reduced or eliminated until the debt is solved.

“Instead of putting more money into prisons, this may sound crazy, but how about taking a bunch of troop ships from World War II, putting them about three miles out to sea and using them as prisons?” one caller said. “It may be considered cruel and unusual punishment, but a lot of us did it in World War II and it wasn't cruel then, and it would save a lot of money.”

A woman suggested the country default on all of its loans. No one, she suggested, would lend the government money again, forcing Congress and President Clinton to balance the budget for the first time since 1969.

Another caller suggested Congress simply stop paying interest, or a portion of the interest, on the debt until our economy improves.

Morris Langman of Latham suggested partial interest payments resume when unemployment falls to 4 percent and full payments follow when unemployment falls to 2 percent.

“This country has to put people back to work, but it doesn't have the money to put people back to work,” said Langman, suggesting his idea would free up federal money for investment. “Who has the money? The moneylenders. When it comes to who to take it from, who's the best suited to be a little more lenient -- a guy on welfare or Social Security, or a moneylender? Instead of starting like we always do at the tail end, with people who can afford it least, how about for once starting at the top?

“I'm not talking about reneging on the debt. I'm talking about postponing the payments and linking it to unemployment.”

Despite the range of sometimes conflicting ideas, one point was consistent: The debt remains a point of embarrassment for many people, callers said, symbolizing a Washington establishment undisciplined and unwilling to police itself.

“The first goal should be to eliminate the debt, not just reduce the deficit. Number two, we should elect politicians whose number one goal is to eliminate the debt,” a caller said. “If we don't do this, we are doomed.”

Two people said with the Cold War over the country has a rare opportunity to cut into a debt which could cripple the economy for generations. Military spending should be slashed more and resources focused on converting military industries into consumer-product manufacturers.

Some areas were off-limits to cutbacks, however. Reduce the deficit, but don't back on fighting crime, some callers said. Reduce welfare, Medicaid, or Social Security payments instead, especially to older recipients above certain income levels.

Another suggested Social Security payments begin later and he subject to taxes much like federal pensions.

But Mary Louise Adams of Schenectady urged the generations not to fight each other and to focus on Washington's spending practices. Foreign aid we cannot afford to send to Russia and elsewhere should be eliminated, she said. Japan, she noted, is fighting trade relations even after our post-World War II efforts to rebuild that country. Finally, government officials pay raises should be rescinded.

One woman suggested a one percent national sales tax devoted entirely to deficit reduction. Another caller supported eliminating the federal departments of energy, education and agriculture, along with agricultural subsidies.

“If we cut the government, the size of this enormous bureaucracy, in half, and cut the budget in half, we can then look at what’s left that we need to do and whether we have money to reduce taxes and cut the deficit or not,” he said.

Still another caller repeated a timeworn complaint: Americans too often spend money on foreign-made products instead of domestic items. Keeping money in the United States keeps jobs here, he argued.

“If we continue to buy foreign products and have an unbalanced trade deficit,” the caller said, “what happens is we no longer have money in this country to generate jobs, generate tax revenues and support programs.